Faculty Of Graduate Studies

Mathematics Program

DYNAMICS AND BIFURCATION OF SECOND ORDER RATIONAL DIFFERENCE EQUATION WITH QUADRATIC TERMS.

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M.Sc.Thesis

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الإهداء

إلى من علمتني الأحرف والكلمات الروح الطاهرة (أمي رحمها الله)
إلى سندي في هذه الحياة (أبي الغالي)
إلى من هم نور حياتي (زوجة أبي وأخوتي وأخواتي)
إلى من أرشدتني إلى السير على نهج القرآن (عمتي العزيزة)
إلى من كانوا لي دعماً (أهلي وأحبتي)
إلى من لا نكهة للحياة بدونهن (صديقاتي)
إلى كل من كانوا معي في مسيرتي العلمية، وكل من كان له دور في وصولي لهذا اليوم
وأخص بالذكر (الدكتور محمد صالح)
والشكر موصولٌ إلى لجنة النقاش (الدكتور مروان العقيلي والدكتور علاء تلاحمة)
إلى المعلم الأول (سيدنا محمد صلى الله عليه وسلم)
إلى وطني (فلسطين)

كله خالصاً لوجه الله تعالى

Abstract

This thesis aims mainly to study some results concerning dynamics and bifurcation of two special cases of second order rational difference equations with quadratic terms. And introduce some Matlab codes that use thesis results.
We consider the second order, quadratic rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n^2 + Cx_{n-1}}, \; n = 0, \; 1, \; 2, \; ...$$

and

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}^2}{A + Bx_n + Cx_{n-1}^2}, \; n = 0, \; 1, \; 2, \; ...$$

with positive parameters $\alpha, \; \beta, \; A, \; B, \; C$, and non-negative initial conditions.

We investigate local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points. And we study the types of bifurcation exist where the change of stability occurs. Then, we give some Matlab codes that use thesis results and numerical discussions with figures to support our results.

الملخص

هذه الرسالة تهدف بشكل أساسي إلى دراسة بعض النتائج المتعلقة بالخصائص الديناميكية و التشعبات لحاتين خاصتين من المعادلات التفاضلية المنفصلة النسبية من الدرجة الثانية مع حدود تربيعية. و تقديم بعض النصوص البرمجية باستخدام الماتلاب التي تستخدم نتائج هذه الرسالة.

لقد درسنا المعادلتين النسبيتين التاليتين

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n^2 + Cx_{n-1}},\ n = 0,\ 1,\ 2,\ ...$$

و

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}^2}{A + Bx_n + Cx_{n-1}^2},\ n = 0,\ 1,\ 2,\ ...$$

بتخصيص المعاملات الموجبة، والنقاط الابتدائية الغير سالبة. سوف نفحص الثبات المحلي، الفترات غير المختلفة، محدودية الحلول الموجبة، وتحليل أنصاف الدورات، والثبات الشامل للنقاط الثابتة الموجبة. و سوف ندرس أنواع التشعبات المتواجدة عند تغير الثبات. ثم نعطي بعض النصوص البرمجية التي تستخدم نتائج الرسالة والمكتوبة بواسطة برنامج ماتلاب و بعض الأمثلة

العددية مع الرسومات التي تدعم النتائج.

CONTENTS

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0. INTRODUCTION

A dynamical system is a system whose behavior at a given time depends on its behavior at one or more previous time. One of the main objectives in the theory of dynamical systems is the study of the behavior of orbits near fixed points.

Dynamical systems are a fundamental part of bifurcation theory which studies the changes in the qualitative or topological structure of systems. The term bifurcation refers to the phenomenon of a system exhibiting new dynamical behavior as the parameter is varied.

Our Equations ((0.1.2), (0.1.3)) are special cases of equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1} + \gamma x_n + \eta x_{n-1}^2 + \zeta x_n x_{n-1} + \ell x_n^2}{A + D x_{n-1} + B x_n + C x_{n-1}^2 + E x_n x_{n-1} + F x_n^2}, \ n = 0, \ 1, \ 2, \ ... \qquad (0.1.1)$$

Some special cases of (0.1.1) have been considered in many papers. In [2] and [3] global stability character, the periodic nature, and the boundedness of solutions of special cases of equation

$$x_{n+1} = \frac{\alpha + \beta x_n x_{n-1} + \gamma x_{n-1}}{A + B x_n x_{n-1} + C x_{n-1}}, \ n = 0, \ 1, \ 2,$$

have been studied, with non-negative parameters and with arbitrary non-negative initial conditions such that the denominator is always positive.

A. M. Amleh, E. Camouzis and G. Ladas [1] considered equations 24 and 25 in [3], they confirmed some conjectures and solved some open problems stated.

In [7] M. GariT-Demirović et al. investigated global behavior of the equation

$$x_{n+1} = \frac{x_{n-1}^2}{ax_n^2 + bx_n x_{n-1} + Cx_{n-1}^2}, \ n = 0, \ 1, \ 2,$$

where the parameters a, b, and c are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary non-negative numbers such that $x_{-1} + x_0 > 0$.

Global asymptotic stability and Neimark-Sacker bifurcation of the difference equation

$$x_{n+1} = \frac{F}{bx_n x_{n-1} + Cx_{n-1}^2 + f}, \ n = 0, \ 1, \ 2,$$

have been investigated by M. R. S. Kulenović et al. [8], with non-negative parameters and non-negative initial conditions such that the denominator is always positive.

Y. Kostrov and Z. Kudlak in [14] studied the boundedness character, local and global stability of solutions of the following second-order rational difference equation with quadratic denominator,

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{B + Dx_n x_{n-1} + x_{n-1}}, \ n = 0, \ 1, \ 2,$$

where the coefficients are positive numbers, and the initial conditions are non-negative numbers such that the denominator is nonzero.

S. Moranjkić, and Z. Nurkanović [13] investigated local and global dynamics of difference equation

$$x_{n+1} = \frac{Bx_n x_{n-1} + Cx_{n-1}^2 + F}{bx_n x_{n-1} + cx_{n-1}^2 + f}, \ n = 0, \ 1, \ 2,$$

with positive parameters and nonnegative initial conditions.

In [6] the dynamics and behavior of the solution rational difference equation of the form

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n + Cx_{n-1}}, \ n = 0, \ 1, \ 2, \ ...$$

was studied with positive parameters α, β, A, B, C, and non-negative initial conditions. They focus on the dynamic behavior of the positive fixed point and the type of bifurcation exist where the change of stability occurs.

A. Shareef [4] studied Neimark-Sacker bifurcation of higher order rational difference equations.

This thesis consists of four main chapters. In chapter 1, we explain the definition of dynamical systems. Then, we focus on fixed points and their stability of first order and second order discrete dynamical systems. Chapter 2 studies types of bifurcation and their sufficient conditions in simplest forms in discrete dynamical systems of one and two dimensions. Chapter 3 studies the dynamics and behavior of the solutions of the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n^2 + Cx_{n-1}}, \; n = 0, \; 1, \; 2, \; ... \tag{0.1.2}$$

with positive parameters $\alpha, \; \beta, \; A, \; B, \; C$, and non-negative initial conditions.
We focus on local stability, invariant intervals, boundedness of solutions, periodic solutions of prime period two and global stability of positive fixed points. We study also the types of bifurcation exist where the change of stability occurs. Then, we give some Matlab codes that use thesis results and numerical discussions with figures to support our results.

Chapter 4 studies the dynamics and behavior of the solutions of the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}^2}{A + Bx_n + Cx_{n-1}^2}, \; n = 0, \; 1, \; 2, \; ... \tag{0.1.3}$$

with positive parameters $\alpha, \; \beta, \; A, \; B, \; C$, and non-negative initial conditions.
We focus on local stability, invariant intervals, boundedness of solutions, periodic solutions of prime period two and global stability of positive fixed points. We study also the types of bifurcation exist where the change of stability occurs. Then, we give some Matlab codes that use thesis results and numerical discussions with figures to support our results.

1. BASIC DEFINITIONS AND RESULTS

Dynamical systems include state (phase) space and a law of evolution of the state in time. The state space contains points that characterize all possible states of the system. Each point in state space must be sufficient not only to describe the current "position" of the system but also to determine its evolution.

Evolution law is defined as a map f^t for given $t \in T$, and f^t is defined on the phase space X as

$$f^t : X \to X$$

f^t is often called the evolution operator of the dynamical system. f^t transforms an initial state $x_0 \in X$ into some state $x_t \in X$ at time t as follows

$$f^t x_0 = x_t$$

or we denote $f^t x_0$ by $x(t)$.

Definition 1. *[5] A dynamical system is a triple $\{T, X, f^t\}$, where T is a time set, X is a phase space, and $f^t : X \to X$ is a family of evolution operators parametrized by $t \in T$.*

In this thesis we consider dynamical systems with discrete (integer) time, whose law of evolution is difference equation.

1.1 Dynamics of first-order difference equations

Consider the first order difference equation of the form

$$x(n+1) = f(x(n)), \; n = 0, \; 1, \; 2, \; \dots \tag{1.1.1}$$

where $f : X \to X$ is a continuous map, and $x(0)$ is an initial condition. Note that

$$\begin{aligned} x(1) &= f(x(0)) \\ x(2) &= f(x(1)) = f(f(x(0))) \\ &. \\ &. \\ &. \\ x(n+1) &= f(x(n)) = f^n(x(0)). \end{aligned}$$

We can also denote $x(n)$ by x_n.

Definition 2. *[9] A point $\bar{x} \in X$ is an equilibrium (fixed) point of Equation (1.1.1) if $f(\bar{x}) = \bar{x}$.*

In other words, $\bar{x}$ is a constant solution of (1.1.1), since if $x(0) = \bar{x}$ is an initial point, then $x(1) = f(\bar{x}) = \bar{x}$, and $x(2) = f(x(1)) = f(\bar{x}) = \bar{x}$, and so on.

Graphically, if we draw the graph of f, and then we draw the diagonal line $y = x$, the x-coordinate of intersection points are the equilibrium points of f.

Example 1.1. *Consider the function*

$$f(x) = x^3.$$

To find fixed points of f we solve $f(x) = x$. Hence the fixed points are $-1, \; 0, \; 1$. Figure (1.1) illustrate how to find these fixed points.

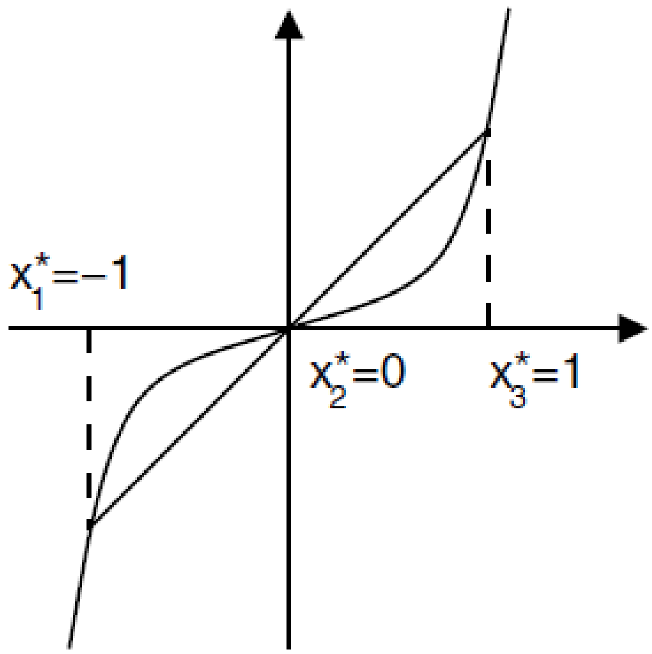


Fig. 1.1: Fixed points of $f(x) = x^3$.

Definition 3. *[5] An orbit starting at* x_0 *is an ordered subset of the state space* X, $Or(x_0) = \{x \in X : \ x = f^t(x_0), \ for\ all\ t \in T\ such\ that\ f^t(x_0)\ is\ defined\}$.

It is possible in difference equations that a solution may not be an equilibrium point but may reach one after finitely many iterations. This leads to the following definition.

Definition 4. *[9] Consider Equation (1.1.1), let* $x^* \in X$. *If there exists a positive integer* k *and an equilibrium point* $\bar{x}$ *of (1.1.1) such that* $f^k(x^*) = \bar{x}$, *and* $f^{k-1}(x^*) \neq \bar{x}$, *then* x^* *is an eventually fixed (equilibrium) point.*

Example 1.2. *Let* T *be the tent map defined by*

$$T(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \frac{1}{2} < x \leq 1 \end{cases}$$

$\frac{1}{8}$ is an eventually fixed point. since $T^4(\frac{1}{8}) = 0$, $T^3(\frac{1}{8}) = 1 \neq 0$, and $x^* = 0$ is a fixed point of T.

Definition 5. *[9] Consider Equation (1.1.1), let $b \in X$. If for some positive integer k, $f^k(b) = b$, then b is a k-periodic point of, and the periodic orbit of b is called a k-cycle.*

1.2 Stability of one-dimensional maps

Analyzing the behavior of a dynamical system solutions near an equilibrium point is one of the main objectives in the study of dynamical system, which constitutes the stability theory. We introduce the definition of stability and some results concerning it.

Definition 6. *[11] Let $\bar{x}$ be an equilibrium point of Equation (1.1.1).*

1. *The equilibrium point is called stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$ for all $n > 0$.*
2. *The equilibrium point is called attracting, if there exists $\gamma > 0$ such that if $|x_0 - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.*

   The equilibrium $\bar{x}$ is called a global attractor if $\gamma = \infty$.
3. *The equilibrium point is called asymptotically stable if it is stable and attracting. If $\gamma = \infty$, $\bar{x}$ is called globally asymptotically stable.*
4. *The equilibrium point is called unstable if it is not stable.*

We have the following results concerning the asymptotic stability of equilibrium points.

Theorem 1.1. *[11] Consider Equation (1.1.1) where f is continuously differentiable at the equilibrium point $\bar{x}$. Then:*

1. *If* $|f'(\bar{x})| < 1$*, then* $\bar{x}$ *is asymptoticly stable.*

2. *If* $|f'(\bar{x})| > 1$*, then* $\bar{x}$ *is unstable.*

Note that the equilibrium point has two types:

1. Hyperbolic If $|f'(\bar{x})| \neq 1$.

2. Non-hyperbolic if $|f'(\bar{x})| = 1$.

Now we introduce some results for non-hyperbolic fixed points.

Theorem 1.2. *[11] Suppose that for an equilibrium point* $\bar{x}$ *of (1.1.1),* $f'(\bar{x}) = 1$*. The following statements then hold:*

1. *If* $f''(\bar{x}) \neq 0$*, then* $\bar{x}$ *is unstable.*

2. *If* $f''(\bar{x}) = 0$ *and* $f'''(\bar{x}) > 0$ *, then* $\bar{x}$ *is unstable.*

3. *If* $f''(\bar{x}) = 0$ *and* $f'''(\bar{x}) < 0$ *, then* $\bar{x}$ *is asymptotically stable.*

Definition 7. *[11] The Schwarzian derivative of a function* f *denoted by* Sf *is defined by*

$$Sf = \frac{f'''(x)}{f'(x)} - \frac{3}{2}[\frac{f''(x)}{f'(x)}]^2.$$

If $f'(\bar{x}) = -1$, then

$$Sf(\bar{x}) = -f'''(\bar{x}) - \frac{3}{2}[f''(\bar{x})]^2.$$

Theorem 1.3. *[11] Suppose that for an equilibrium point* $\bar{x}$ *of (1.1.1),* $f'(\bar{x}) = -1$*. The following statements then hold:*

1. *If* $Sf(\bar{x}) < 0$*, then* $\bar{x}$ *is asymptotically stable.*

2. *If* $Sf(\bar{x}) > 0$*, then* $\bar{x}$ *is unstable.*

Example 1.3. *Consider the function*

$$f(x) = 3x - x^2, \quad x \in [0, 3]$$

To find fixed points of f solve $f(x) = x$. Hence the fixed points are $\bar{x_1} = 0$, $\bar{x_2} = 2$. Now we find f' to determine the stability of $\bar{x_1}$ and $\bar{x_2}$,

$$f'(x) = 3 - 2x.$$

For $\bar{x_1}$

$$f'(0) = 3 > 1,$$

so using Theorem 1.1, $\bar{x_1}$ is unstable.

For $\bar{x_2}$

$$f'(2) = -1$$

so we find $Sf(2)$

$$Sf(2) = -6 < 0$$

using Theorem 1.3, $\bar{x_2}$ is stable.

We can use a graphical method for analyzing the stability of equilibrium points for (1.1.1) called Cobweb diagram. We draw a graph of f in the $(x(n), x(n+1))$ plane. Then, given $x(0) = x_0$, we pinpoint the value $x(1)$ by drawing a vertical line through x_0 so that it also intersects the graph of f at $(x_0, x(1))$. Next, draw a horizontal line from $(x_0, x(1))$ to meet the diagonal line $y = x$ at the point $(x(1), x(1))$. A vertical line drawn from the point $(x(1), x(1))$ will meet the graph of f at the point $(x(1), x(2))$. Continuing this process, we can find $x(n)$ for all $n > 0$.

Example 1.4. *Consider the function in Example 1.2. Cobweb diagram (1.2) shows that $\bar{x_2}$ is stable. Figure (1.3) shows the behavior of x_n near $\bar{x_2}$.*

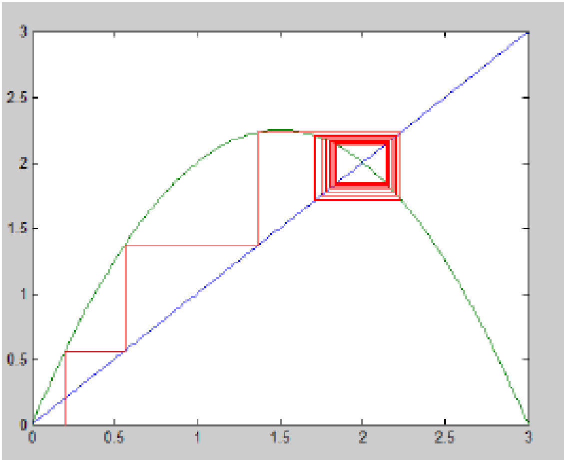


Fig. 1.2: The Cobweb diagram: $\bar{x}_2$ is asymptotically stable fixed point.

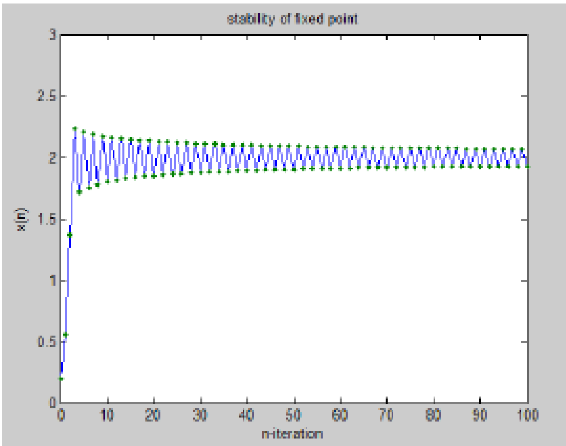


Fig. 1.3: The behavior of the solutions near the fixed point x=2

1.3 Dynamics of second order difference equations

Consider the second order difference equation,

$$x(n+1) = f(x(n), x(n-1)),\ n = 0,\ 1,\ 2,\ ... \tag{1.3.1}$$

Where $f : I \times I \to I$ is a continuously differentiable function, and I is an interval of real numbers. Then for every set of initial conditions $x_{-1},\ x_0 \in I$ the difference equation (1.3.1) has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 8. *[9] A point $\bar{x} \in I$ is an equilibrium point of Equation (1.3.1) if $f(\bar{x}, \bar{x}) = \bar{x}$.*

Definition 9. *[9] Let $\bar{x}$ be an equilibrium point of Equation (1.3.1). Then*

1. *$\bar{x}$ is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$, then $|x_n - \bar{x}| < \epsilon$ for all $n > 0$.*
2. *$\bar{x}$ is called attracting, if there exists $\gamma > 0$ such that if $|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \bar{x}$.*
3. *$\bar{x}$ is called a global attractor if for every $x_{-1},\ x_0 \in I$ we have $\lim_{n\to\infty} x_n = \bar{x}$.*
4. *$\bar{x}$ is called globally asymptotically stable if it is locally stable and a global attractor.*
5. *$\bar{x}$ is called unstable if it is not stable.*

Definition 10. *[9]*

1. *A solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (1.3.1) is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -1$.*
2. *A solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (1.3.1) is said to be periodic with prime period p, or p-cycle if it is periodic with period p and p is the least positive integer for which $x_{n+p} = x_n$ for all $n \geq -1$.*

Definition 11. *[9] Consider the difference equation (1.3.1). Then the linearized equation associated with this difference equation is*

$$y_{n+1} = ay_n + by_{n-1},\ \ n = 0,\ 1,\ 2,\ \ldots$$

Where $a = \frac{\partial f}{\partial u}(\bar{x}, \bar{x})$, and $b = \frac{\partial f}{\partial v}(\bar{x}, \bar{x})$ denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium $\bar{x}$.

And the characteristic equation of (1.3.1) is

$$\lambda^2 - a\lambda - b = 0 \tag{1.3.2}$$

Theorem 1.4. *[11] (Linearized Stability)*
Consider the characteristic Equation (1.3.2).

1. *If both characteristic roots of (1.3.2) lie inside the unit disk in the complex plane, then the equilibrium $\bar{x}$ of (1.3.1) is locally asymptotically stable.*
2. *If at least one characteristic root of (1.3.2) is outside the unit disk in the complex plane, the equilibrium point $\bar{x}$ is unstable.*
3. *If one characteristic root of (1.3.2) is on the unit disk and the other characteristic root is either inside or on the unit disk, then the equilibrium point $\bar{x}$ may be stable, unstable, or asymptotically stable.*
4. *A necessary and sufficient condition for both roots of (1.3.2) to lie inside the unit disk in the complex plane, is*

$$|a| < 1 - b < 2.$$

Let $A = Jf(\bar{x})$ is the Jacobian matrix of f at $\bar{x}$, where

$$Jf(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} |_{\bar{x}}$$

Let $\rho(A) = \max_i\{|\lambda_i|, \lambda_i \textit{is an eigenvalue of } A\}$ be the spectral norm of A.

Theorem 1.5. *[10] Consider the map $f : H \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map, where H is an open subset of $\mathbb{R}^2$, $\bar{x}$ is a fixed point of f, $A = Jf(\bar{x})$, with spectral norm $\rho(A)$. Then the following statements hold:*

1. *If $\rho(A) < 1$, then $\bar{x}$ is asymptotically stable.*

2. *If* $\rho(A) > 1$, *then* $\bar{x}$ *is unstable.*

3. *If* $\rho(A) = 1$, *then* $\bar{x}$ *may or may not be stable.*

Another important way to determine the stability of fixed points is given in the following result.

Theorem 1.6. *[10] Consider the map* $f : H \subset \mathbb{R}^2 \to \mathbb{R}^2$, *and let* $A = Jf(\bar{x})$, *with spectral norm* $\rho(A)$. *Then* $\rho(A) < 1$, *if and only if*

$$|tr(A)| - 1 < det(A) < 1$$

where $tr(A)$ *is the trace of* A, *and* $det(A)$ *is the determinant of* A.

Note that the stability region in the trace-determinant plane is enclosed by the lines $det(A) = tr(A) - 1$, $det(A) = -tr(A) - 1$, and $det(A) = 1$. These lines are important in the study of the bifurcation of any second order dimensional system.

The following theorem will be used to investigate global stability of fixed points.

Theorem 1.7. *[9] Let* $[a, b]$ *be an interval of real numbers and assume that* $f : [a, b] \times [a, b] \to [a, b]$ *is a continuous function satisfying the following properties:*

1. $f(x, y)$ *is non-increasing in* $x \in [a, b]$ *for each* $y \in [a, b]$, $f(x, y)$ *is non-decreasing in* $y \in [a, b]$ *for each* $x \in [a, b]$.

2. *The difference Equation (1.3.1) has no solutions of prime period two in* $[a, b]$.

   Then (1.3.1) has a unique equilibrium $\bar{x} \in [a, b]$ *and every solution of (1.3.1) converges to* $\bar{x}$.

2. BIFURCATION OF FIXED POINTS

The expression "bifurcation" is extremely general. We use it to describe the orbit structure near non-hyperbolic fixed points.

Definition 12. *Bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value.*

Bifurcation diagram display the location and stability of fixed point as a function of the parameter in a single plot. The locations of unstable fixed points are shown dashed, while stable fixed points are represented by solid lines.

For one-dimensional systems, there are several types of bifurcations which are saddle-node, transcritical, pitchfork and period-doubling bifurcation.

2.1 Bifurcation of one-parameter family of one-dimensional maps

Consider a one-parameter map

$$x \to f(x, \mu), \;\; x \in \mathbb{R}, \;\; \mu \in \mathbb{R}$$

a fixed point $(\bar{x}, \mu^*)$ is a bifurcation point if either only one branch or more than one branch of fixed points passes through $(\bar{x}, \mu^*)$ in the $x - \mu$ plane, then it lies entirely on one side of the line $\mu = \mu^*$ in the $x - \mu$ plane.

In this section we present general conditions under which a one-parameter family of one-dimensional map will undergo a saddle-node, pitchfork, transcritical and period-doubling bifurcation.

Note that if we have more than one parameter, we will fix all parameters except one.

2.2 The Saddle-node bifurcation

Saddle-node bifurcation associated with the appearance of a slope 1. A unique curve of fixed points passes through the non hyperbolic fixed point $(\bar{x}, \mu^*)$. Moreover, the curve lies entirely on one side of the $\mu = \mu^*$ in the $x - \mu$ plane.

Theorem 2.1 (The Saddle-node Bifurcation). *[10]*
Suppose that $f(x, \mu)$ is a C^2 one-parameter family of one-dimensional maps (i.e., both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial \mu^2}$ exist and are continuous), and $\bar{x}$ is a fixed point of $f(x, \mu)$, with $\frac{\partial f}{\partial x}(\bar{x}, \mu^*) = 1$. Assume also that

$$A = \frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) \neq 0.$$

And

$$B = \frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*) \neq 0.$$

Then there exists an interval I around $\bar{x}$ and a C^2 map $\mu = p(x)$, where $p : I \to \mathbb{R}$ such that $p(\bar{x}) = \mu^*$, and $f(x, p(x)) = x$.
Moreover, if $AB < 0$, the fixed points exist for $\mu > \mu^*$, and, if $AB > 0$, the fixed points exist for $\mu < \mu^*$.

To prove Theorem 2.1, we need the Implicit Function Theorem.

Theorem 2.2. *Suppose that $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 map in both variables such that for some $(\mu^*, x^*) \in \mathbb{R} \times \mathbb{R}$, $G(\mu^*, x^*) = 0$ and $\frac{\partial G}{\partial \mu}(\mu^*, x^*) \neq 0$. Then, there exists an open interval J around μ^*, and open interval I around x^*, and a C^1 map $\mu = p(x)$, where $p : I \to J$ such that*

1. *$p(x^*) = \mu^*$.*

2. $G(p(x), x) = 0$, *for all* $x \in I$.

Proof of Theorem 2.1. Suppose that $f(x, \mu)$ is a C^2 one-parameter family of one-dimensional maps (i.e., both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial \mu^2}$ exist and are continuous), and $\bar{x}$ is a fixed point of $f(x, \mu)$, with $\frac{\partial f}{\partial x}(\bar{x}, \mu^*) = 1$. Assume also that $A = \frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) \neq 0$ and $B = \frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*) \neq 0$.
Let

$$G(x, \mu) = f(x, \mu) - x.$$

Note that G is a C^1 map and $G(\bar{x}, \mu^*) = 0$ and

$$\frac{\partial G}{\partial \mu}(\bar{x}, \mu^*) = \frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) = A \neq 0.$$

Applying the implicit function theorem, there exist an open interval J around $\bar{x}$, and open interval I around μ^*, and a C^1 map $\mu = p(x)$, where $p : J \to I$ such that

$$p(\bar{x}) = \mu^*$$

and

$$G(x, p(x)) = 0, \ \textit{for all } x \in J.$$

Thus

$$f(x, p(x)) = x, \ \textit{for all } x \in J. \tag{2.2.1}$$

Differentiating both sides of the last equation with respect to x, we obtain

$$\frac{\partial f}{\partial x}(\bar{x}, \mu^*) + \frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) p'(\bar{x}) = 1.$$

Since $\frac{\partial f}{\partial x}(\bar{x}, \mu^*) = 1$ and $\frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) \neq 0$, we have $p'(\bar{x}) = 0$.

Differentiating (2.2.1) one more time with respect to x we get

$$\frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*) + \frac{\partial^2 f}{\partial \mu^2}(\bar{x}, \mu^*) p''(\bar{x}) = 0.$$

This implies

$$p''(\bar{x}) = -\frac{\frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*)}{\frac{\partial^2 f}{\partial \mu^2}(\bar{x}, \mu^*)} = -\frac{B}{A}.$$

So the function $p(x) = \mu$ has critical point at $x = \bar{x}$. If $AB < 0$, then $p''(\bar{x}) > 0$ and the curve $p(x)$ is concave upward at $x = \bar{x}$. Hence, the curve $p(x)$ opens to the right and if $AB > 0$, then $p(x)$ is concave downward at $x = \bar{x}$. Figures (2.1) and (2.2) illustrate saddle-node bifurcation when $AB < 0$ and $AB > 0$.

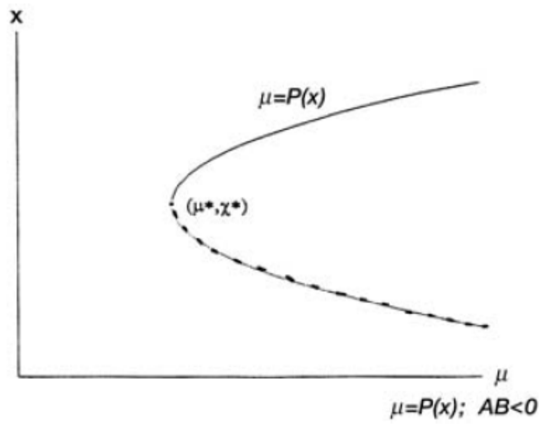


Fig. 2.1: Saddle-node bifurcation when $AB < 0$.

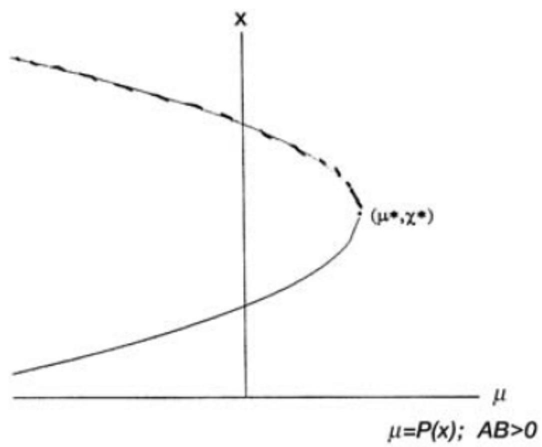


Fig. 2.2: Saddle-node bifurcation when $AB > 0$.

■

Example 2.1. *Consider the map*

$$f(x, \mu) = \mu - x^2, \;\; x \in \mathbb{R}, \;\; \mu \in \mathbb{R}.$$

To find the fixed points of $f(x, \mu)$ *we solve the following equation*

$$x^2 + x - \mu = 0,$$

so the fixed points are

$$\bar{x}_{1,2} = \frac{-1 \pm \sqrt{1+4\mu}}{2}.$$

Note that the fixed points exist for $\mu \geq -\frac{1}{4}$*. And we have a non-hyperbolic fixed point* $\bar{x} = -\frac{1}{2}$ *with* $\frac{\partial f}{\partial x}(\bar{x}, \mu^*) = 1$ *when* $\mu^* = -\frac{1}{4}$*.*

Observe that $\frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) \neq 0$*, and* $\frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*) \neq 0$*. Using* ***Theorem 2.1*** *the saddle-node bifurcation is present at* $(-\frac{1}{2}, -\frac{1}{4})$*.*

In order to draw the bifurcation diagram we check the stability of the system near the bifurcation point $(-\frac{1}{2}, -\frac{1}{4})$*.*
Note that

$$f'(x, \mu) = -2x.$$

The upper branch $x = -\frac{1-\sqrt{1+4\mu}}{2}$ *is asymptotically stable if*

$$|f'(-\frac{1-\sqrt{1+4\mu}}{2}, \mu)| < 1$$

which holds if $-\frac{1}{4} < \mu < \frac{3}{4}$*. So the upper branch is asymptotically stable if* $-\frac{1}{4} < \mu < \frac{3}{4}$*. See figure 2.3.*

The lower branch is unstable since

$$f'(-\frac{1+\sqrt{1+4\mu}}{2}, \mu) = 1 + \sqrt{1+4\mu} > 1$$

for all values of μ*.*

2.3 Transcritical bifurcation

Consider a one-parameter map

$$x \to f(x, \mu), \ x \in \mathbb{R}, \ \mu \in \mathbb{R}$$

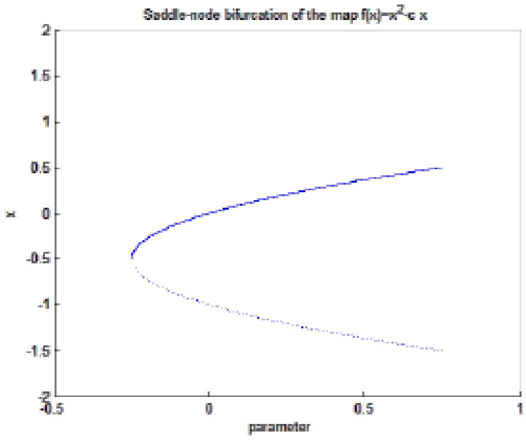


Fig. 2.3: Saddle-node bifurcation.

with fixed point $\bar{x}$. Transcritical bifurcation is another type of bifurcation of one-dimensional maps. This type appears when we have two curves of fixed points that intersect at the non-hyperbolic fixed point $(\bar{x}, \mu^*)$ in the $\mu - x$ plane. Both curves exist on either sides of the line $\mu = \mu^*$. However, the stability of the fixed point along a given curve changes when passing through $\mu = \mu^*$.

Theorem 2.3. *[12] Suppose that $f(x,\mu)$ is a $C^r(r > 1)$ map where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ and $(\bar{x}, \mu^*)$ is a non-hyperbolic fixed point of $f(x,\mu)$ such that*

$$\frac{\partial f}{\partial x}(\bar{x}, \mu^*) = 1,$$
$$\frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) = 0,$$
$$\frac{\partial^2 f}{\partial x \partial \mu}(\bar{x}, \mu^*) \neq 0,$$

and

$$\frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*) \neq 0,$$

Then f undergoes a transcritical bifurcation at $(\bar{x}, \mu^*)$.

Proof: Let

$$G(x,\mu) = f(x,\mu) - x$$

since

$$\frac{\partial G}{\partial \mu}(\bar{x}, \mu^*) = \frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) = 0,$$

we can not apply the Implicit Function Theorem. So we use the following function:

$$B(x, \mu) = \begin{cases} \frac{G(x,\mu)}{x-\bar{x}} & if x \neq \bar{x} \\ \frac{\partial G}{\partial x}(\bar{x}, \mu) & if x = \bar{x} \end{cases}$$

We have

$$B(\bar{x}, \mu^*) = \frac{\partial G}{\partial x}(\bar{x}, \mu^*) = \frac{\partial f}{\partial x}(\bar{x}, \mu^*) - 1 = 0$$

and

$$\frac{\partial B}{\partial \mu}(\bar{x}, \mu^*) = \frac{\partial}{\partial \mu}(\frac{\partial G}{\partial x}(\bar{x}, \mu^*)) = \frac{\partial^2 f}{\partial \mu \partial x}(\bar{x}, \mu^*) \neq 0.$$

By Implicit Function Theorem there is a C^1 map $\mu = p(x)$ defined on an interval I around $\bar{x}$ such that $p(\bar{x}) = \mu^*$ and

$$B(x, p(x)) = 0, \ \forall x \in I. \tag{2.3.1}$$

Hence,

$$\frac{G(x, \mu)}{x - \bar{x}} = 0$$

for $x \neq \bar{x}$, so $f(x, p(x)) = x$. Differentiate (2.3.1) with respect to x we get

$$\frac{\partial B}{\partial x}(\bar{x}, \mu^*) + \frac{\partial B}{\partial \mu}(\bar{x}, \mu^*)p'(\bar{x}) = 0$$

since

$$\frac{\partial B}{\partial x}(\bar{x}, \mu^*) = \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\bar{x}, \mu^*) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*) \neq 0$$

and

$$\frac{\partial B}{\partial \mu}(\bar{x}, \mu^*) = \frac{\partial^2 G}{\partial \mu \partial x}(\bar{x}, \mu^*) \neq 0,$$

we have $p'(\bar{x}) \neq 0$. This means that $p(x)$ does not coincide with $x = \bar{x}$ and exists on both sides of $\mu = \mu^*$. ■

Example 2.2. *Consider the map*

$$f(x, \mu) = \mu x + x^2, \ x \in \mathbb{R}, \ \mu \in \mathbb{R}.$$

To find the fixed points of $f(x,\mu)$ *we solve the following equation*

$$x^2 + (\mu - 1)x = 0,$$

so the fixed points are

$$\bar{x}_1 = 0, \quad \bar{x}_2 = 1 - \mu$$

for $\mu \neq 0$.

Since $\frac{\partial f}{\partial x}(0,1) = 1$, $(0,1)$ *is a non-hyperbolic fixed point of the map* f.

To check the stability of the fixed points near the point $(0,1)$ *we find when*

$$|\frac{\partial f}{\partial x}(0,\mu)| < 1$$

and

$$|\frac{\partial f}{\partial x}(1-\mu,\mu)| < 1.$$

The first inequality holds if $-1 < \mu < 1$, *and the second inequality holds if* $1 < \mu < 3$.

So the branch $x = 0$ *is asymptotically stable if* $-1 < \mu < 1$, *and the branch* $\mu - 1$ *is asymptotically stable if* $1 < \mu < 3$. *The two branches intersect at the bifurcation point* $(0,1)$ *where the branch* $x = 0$ *is stable and the other branch* $\mu - 1$ *is unstable before* $(0,1)$. *Beyond* $\mu = 1$ *the branch* $x = 0$ *becomes unstable and the other branch becomes stable. So change of stability occurs at* $\mu = 1$.

2.4 Pitchfork bifurcation

Consider the one-parameter map

$$x \to f(x,\mu), \; x \in \mathbb{R}, \; \mu \in \mathbb{R}$$

with fixed point $\bar{x}$. Pitchfork bifurcation is a type of bifurcation in one-dimensional systems which appears when we have two curves of fixed points intersect at the non-hyperbolic fixed point $(\bar{x},\mu^*)$ in the $\mu - x$ plane. only one curve exists on both sides

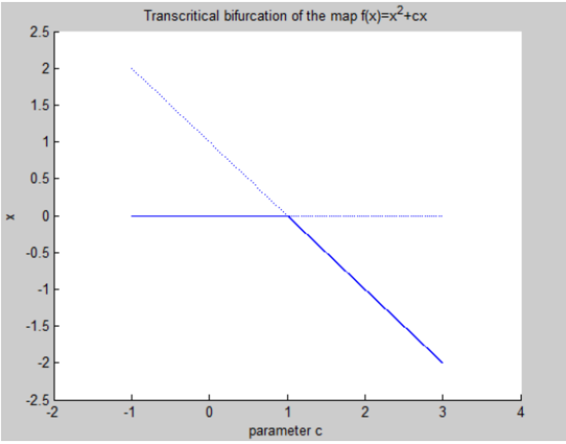


Fig. 2.4: Transcritical bifurcation.

of $\mu = \mu^*$; however, its stability changes when passing through $\mu = \mu^*$. The other curve of fixed point lies entirely in one side of the line $\mu = \mu^*$ and has a stability type that is the opposite of the other curve.

Theorem 2.4. *[12] Suppose that $f(x, \mu)$ is a C^2 one-parameter family of one-dimensional map, where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ with a non-hyperbolic fixed point $(\bar{x}, \mu^*)$ such that*

$$f(\bar{x}, \mu^*) = \bar{x} \text{ and } \frac{\partial f}{\partial x}(\bar{x}, \mu^*) = 1.$$

Assume also that

$$A = \frac{\partial f}{\partial \mu}(\bar{x}, \mu^*) = 0,$$
$$C = \frac{\partial^2 f}{\partial x^2}(\bar{x}, \mu^*) = 0,$$
$$D = \frac{\partial^2 f}{\partial x \partial \mu}(\bar{x}, \mu^*) \neq 0,$$

and

$$E = \frac{\partial^3 f}{\partial x^3}(\bar{x}, \mu^*) \neq 0,$$

Then pitchfork bifurcation is present at $(\bar{x}, \mu^*)$.

Proof: Let

$$G(x,\mu) = f(x,\mu) - x,$$

we have $G(\bar{x},\mu^*) = 0$, and $\frac{\partial f}{\partial x}(\bar{x},\mu^*) = 0$. So we can not apply the Implicit Function Theorem.

Let

$$B(x,\mu) = \begin{cases} \frac{G(x,\mu)}{x-\bar{x}} & if x \neq \bar{x} \\ \frac{\partial G}{\partial x}(\bar{x},\mu) & if x = \bar{x} \end{cases}$$

We have

$$B(\bar{x},\mu^*) = \frac{\partial G}{\partial x}(\bar{x},\mu^*) = \frac{\partial f}{\partial x}(\bar{x},\mu^*) - 1 = 0$$

and

$$\frac{\partial B}{\partial \mu}(\bar{x},\mu^*) = \frac{\partial}{\partial \mu}(\frac{\partial G}{\partial x}(\bar{x},\mu^*)) = \frac{\partial^2 f}{\partial \mu \partial x}(\bar{x},\mu^*) \neq 0.$$

By the Implicit Function Theorem there is a C^1 map $\mu = p(x)$ defined on an interval I around $\bar{x}$ such that $p(\bar{x}) = \mu^*$ and

$$B(x,p(x)) = 0, \ \forall x \in I. \quad (2.4.1)$$

Differentiate (2.4.1) with respect to x we get

$$\frac{\partial B}{\partial x}(\bar{x},\mu^*) + \frac{\partial B}{\partial \mu}(\bar{x},\mu^*)p'(\bar{x}) = 0$$

since

$$\frac{\partial B}{\partial x}(\bar{x},\mu^*) = \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(\bar{x},\mu^*) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\bar{x},\mu^*) = 0$$

and

$$\frac{\partial B}{\partial \mu}(\bar{x},\mu^*) = \frac{\partial^2 G}{\partial \mu \partial x}(\bar{x},\mu^*) \neq 0,$$

we have $p'(\bar{x}) = 0$. So $\bar{x}$ is a critical point of the map $\mu = p(x)$.
Differentiate (2.3.1) again with respect to x, and substitute $(\bar{x},\mu^*)$ and $p'(\bar{x}) = 0$ we get

$$\frac{\partial^2 B}{\partial x^2}(\bar{x},\mu^*) + \frac{\partial B}{\partial \mu}(\bar{x},\mu^*)p''(\bar{x}) = 0.$$

Since

$$\frac{\partial^2 B}{\partial x^2}(\bar{x},\mu^*) = \frac{1}{3}\frac{\partial^3 G}{\partial x^3}(\bar{x},\mu^*) = \frac{1}{3}\frac{\partial^3 f}{\partial x^3}(\bar{x},\mu^*) \neq 0$$

and

$$\frac{\partial B}{\partial \mu}(\bar{x},\mu^*) = \frac{\partial}{\partial \mu}\frac{\partial G}{\partial x}(\bar{x},\mu^*) = \frac{\partial^2 f}{\partial \mu \partial x}(\bar{x},\mu^*) \neq 0$$

$$p''(\bar{x}) = \frac{E}{D} \neq 0.$$

The last formula implies that if $ED < 0$, then $p''(\bar{x}) > 0$ and the curve $p(x)$ is concave upward at $x = \bar{x}$ and if $ED > 0$, then the curve $p(x)$ is concave downward at $x = \bar{x}$. ■

Example 2.3. *Consider the map*

$$f(x,\mu) = \mu x - 2x^3, \;\; x \in \mathbb{R}, \;\; \mu \in \mathbb{R}.$$

To find the fixed points of $f(x,\mu)$, we solve the following equation

$$(\mu - 1)x - 2x^3 = 0,$$

so we have two curves of fixed points

$$x = 0, \quad x^2 = \frac{\mu - 1}{2}.$$

Note that $(0,1)$ is non-hyperbolic fixed point of f such that

$$\frac{\partial f}{\partial x}(0,1) = 1,$$
$$\frac{\partial f}{\partial \mu}(0,1) = 0,$$
$$\frac{\partial^2 f}{\partial x^2}(0,1) = 0,$$
$$\frac{\partial^2 f}{\partial x \partial \mu}(0,1) = 1,$$

and

$$\frac{\partial^3 f}{\partial x^3}(0,1) = -12,$$

So at $(0,1)$ pitchfork bifurcation is present. Now we study the behavior of the system near the bifurcation point $(0,1)$.
$|f'(0,\mu)| < 1$ if $-1 < \mu < 1$ and $|f'(\pm\sqrt{\frac{\mu-1}{2}},\mu)| < 1$ if $1 < \mu < 2$. So we have one branch of stable fixed points $x = 0$ for $-1 < \mu < 1$. Beyond $\mu = 1$ this fixed point looses its stability and two stable branches $x = \pm\sqrt{\frac{\mu-1}{2}}$ appear. Beyond $\mu = 2$ these two branches looses their stability.

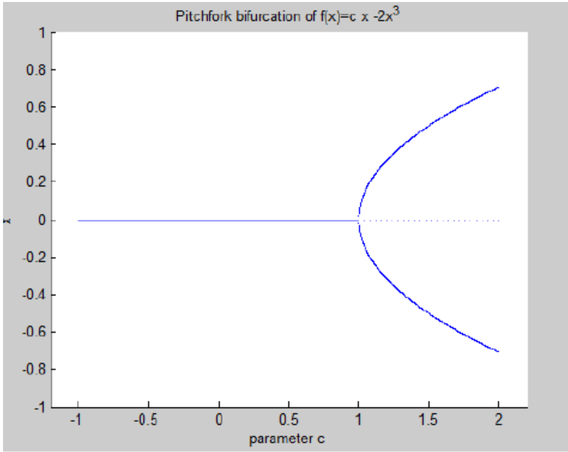


Fig. 2.5: Pitchfork bifurcation.

2.5 Period-Doubling bifurcation

Consider the one-parameter map

$$x \to f(x, \mu), \;\; x \in \mathbb{R}, \;\; \mu \in \mathbb{R}$$

with non-hyperbolic fixed point $\bar{x}$. period-doubling bifurcation is a type of bifurcation in one-dimensional maps which have a non-hyperbolic fixed point $(\bar{x}, \mu^*)$ with slope -1 and the second iterate of the map undergoes pitchfork bifurcation at the same non-hyperbolic fixed point.

Theorem 2.5 (Period-Doubling Bifurcation). *[10] Suppose that $f(x, \mu)$ is a $C^r(r > 2)$ map where $x \in \mathbb{R}, \;\; \mu \in \mathbb{R}$ and $\bar{x}$ is a fixed point of $f(x, \mu)$ such that*

$$\frac{\partial f}{\partial x}(\bar{x}, \mu^*) = -1, \tag{2.5.1}$$

and

$$\frac{\partial^2 f^2}{\partial \mu \partial x}(\bar{x}, \mu^*) \neq 0, \tag{2.5.2}$$

Then there is an interval I around $\bar{x}$ and a function $p : I \to \mathbb{R}$ such that $f(x, p(x)) \neq x$ but $f^2(x, p(x)) = x$.

Proof: Let $f(x,\mu)$ is a C^2 map where $x \in \mathbb{R},\ \mu \in \mathbb{R}$ that satisfies (2.5.1) and (2.5.2). Let

$$G(x,\mu) = f^2(x,\mu) - x$$

where $f^2(x,\mu) = f(f(x,\mu),\mu)$. Note that

$$\frac{\partial G}{\partial \mu}(\bar{x},\mu^*) = \frac{\partial f}{\partial x}(\bar{x},\mu^*)\frac{\partial f}{\partial \mu}(\bar{x},\mu^*) + \frac{\partial f}{\partial \mu}(\bar{x},\mu^*) = 0$$

since (2.5.1) holds. So we can not use the Implicit Function Theorem.

Define the function

$$B(x,\mu) = \begin{cases} \frac{G(x,\mu)}{x-\bar{x}} & if x \neq \bar{x} \\ \frac{\partial G}{\partial x}(\bar{x},\mu) & if x = \bar{x} \end{cases}$$

We have

$$B(\bar{x},\mu^*) = \frac{\partial G}{\partial x}(\bar{x},\mu^*) = [\frac{\partial f}{\partial x}(\bar{x},\mu^*)]^2 - 1 = 0$$

and

$$\frac{\partial B}{\partial \mu}(\bar{x},\mu^*) = \frac{\partial}{\partial \mu}(\frac{\partial G}{\partial x}(\bar{x},\mu))|_{\mu^*} = \frac{\partial^2 f^2}{\partial \mu \partial x}(\bar{x},\mu^*) \neq 0.$$

By Implicit Function Theorem there is a C^1 map $\mu = p(x)$ defined on an interval I around $\bar{x}$ such that $p(\bar{x}) = \mu^*$ and

$$B(x,p(x)) = 0,\ \forall x \in I. \tag{2.5.3}$$

And so

$$\frac{G(x,\mu)}{x-\bar{x}} = 0$$
$$f^2(x,p(x)) = x,$$

so x is a two-period point of $f(x,\mu)$. ■

Example 2.4. *Consider the map*

$$f(x,\mu) = 2x^3 + x - \mu x,\ x \in \mathbb{R},\ \mu \in \mathbb{R}.$$

To find the fixed points of $f(x,\mu)$ we solve the following equation

$$2x^3 - \mu x = 0,$$

so we have two curves of fixed points

$$x = 0, \quad x^2 = \frac{\mu}{2}.$$

Since $f(0,2) = 0$ *and* $\frac{\partial f}{\partial x}(0,2) = -1$, $(0,2)$ *is a non-hyperbolic fixed point of the map* f.

The curve $x = 0$ *is stable if* $0 < \mu < 2$ *and the curve* $x^2 = \frac{\mu}{2}$ *does not exist if* $\mu < 0$ *and unstable for* $\mu \geq 0$.

Thus for $\mu > 2$, *the map has three unstable curves of fixed points. Period-doubling bifurcation may be present at* $(0,2)$. *The two-periodic points are the roots of the function*

$$g(x,\mu) = f^2(x,\mu) - x,$$

where

$$f^2(x,\mu) = (\mu-1)^2 x - 2(\mu^3 - 3\mu^2 + 4\mu - 2)x^3 + O(x^4).$$

Observe that

$$\begin{aligned} f^2(0,2) &= 0, \\ \frac{\partial f^2}{\partial x}(0,2) &= 1, \\ \frac{\partial f^2}{\partial \mu}(0,2) &= 0, \\ \frac{\partial^2 f^2}{\partial x^2}(0,2) &= 0, \\ \frac{\partial^2 f^2}{\partial x \partial \mu}(0,2) &\neq 0, \end{aligned}$$

and

$$\frac{\partial^3 f^2}{\partial x^3}(0,2) \neq 0,$$

Thus $(0,2)$ *is a non-hyperbolic fixed point of the map* $f^2(x,\mu)$ *where this map undergoes a pitchfork bifurcation. So period-doubling bifurcation is present at* $(0,2)$.

We can use the normal form of flip bifurcation theorem to check if the system undergoes a period-doubling (flip) bifurcation. We will study the normal form theorem for flip bifurcation in the simplest form.

2.5.1 The normal form of the period-doubling (flip) bifurcation

Consider the following one-dimensional dynamical system depending on one parameter:

$$x \mapsto -(1+\mu)x + x^3 \equiv f(x,\mu).$$

The map $f(x,\mu)$ is invertible for small $|\mu|$ in a neighborhood of the origin. $f(x,\mu)$ has the fixed point $x_0 = 0$ for all μ with eigenvalue $\lambda = -(1+\mu)$. The point is linearly stable for small $\mu < 0$ and is linearly unstable for $\mu > 0$. And at $\mu = 0$, $\lambda = f_x(0,0) = -1$ so the point is not-hyperbolic. There are no other fixed points near the origin for small $|\mu|$.

Consider now the second iterate $f^2(x,\mu)$. If $y = f(x,\mu)$, then

$$\begin{aligned} f^2(x,\mu) &= f(y,\mu) = -(1+\mu)y + y^3 \\ &= -(1+\mu)(-(1+\mu)x + x^3) + (-(1+\mu)x + x^3)^3 \\ &= (1+\mu)^2 x + [(1+\mu)(2+2\mu+\mu^2)]x^3 + O(x^5). \end{aligned}$$

So the map $f^2(x,\mu)$ has the trivial fixed point $x_0 = 0$. It also has two nontrivial fixed points for small $\mu > 0$, which are

$$x_{1,2} = f^2(x_{1,2},\mu),$$

where $x_{1,2} = \pm(\sqrt{\mu} + O(\mu))$. These two points are stable and constitute a cycle of period two for the original map $f(x,\mu)$ as follows

$$x_1 = f(x_2,\mu),\ x_2 = f(x_1,\mu).$$

As μ approaches zero, the period-two cycle shrinks and disappears. This is a flip (period-doubling) bifurcation and in this case it is called supercritical. Note that the trivial fixed point is stable for $\mu < 0$ and the period-two cycle $\{x_1, x_2\}$ exists for $\mu > 0$.

The case

$$x \mapsto -(1+\mu)x + x^3$$

can be treated in the same way and the flip bifurcation in this case is called subcritical. [5]

2.5.2 Generic flip bifurcation

Theorem 2.6. *[5] Suppose that a one-dimensional map*

$$x \mapsto f(x,\mu), \;\; x \in \mathbb{R}, \;\; \mu \in \mathbb{R}$$

with smooth f, has at $\mu = 0$ a trivial fixed point $x_0 = 0$ and let $\lambda = f_x(0,0) = -1$. Assume the following nondegeneracy conditions are satisfied:

1. $\frac{1}{2}(f_{xx}(0,0))^2 + \frac{1}{3}(f_{xxx}(0,0)) \neq 0.$
2. $f_{x\mu}(0,0) \neq 0.$

Then there are smooth invertible coordinate and parameter changes transforming the system into

$$\xi \mapsto -(1+\alpha)\xi + \xi^3 + O(\xi^4).$$

Proof: By the Implicit Function Theorem,the system has a unique fixed point $x_0(\mu)$ in some neighborhood of the origin for all sufficiently small $|\mu|$, since $f_x(0,0) \neq 1$. We can perform a coordinate shift, placing this fixed point at the origin. Therefore, we can assume without loss of generality that $x = 0$ is the fixed point of the system for $|\mu|$ sufficiently small. Thus, the map f can be written as follows:

$$f(x,\mu) = f_x(0,\mu)x + \frac{1}{2}f_{xx}(0,\mu)x^2 + \frac{1}{6}f_{xxx}(0,\mu)x^3 + O(x^4). \qquad (2.5.4)$$

Where $f_x(0,\mu) = -[1+g(\mu)]$ for some smooth function g. Since $g(0) = 0$ and

$$g'(0) = f_{x\mu}(0,0) \neq 0,$$

the function g is locally invertible and can be used to introduce a new parameter:

$$\alpha = g(\mu).$$

Map (2.5.4) can be written as

$$\tilde{x} = \mu(\alpha)x + a(\alpha)x^2 + b(\alpha)x^3 + O(x^4),$$

where $\mu(\alpha) = -(1+\alpha)$, and the functions $a(\alpha)$ and $b(\alpha)$ are smooth and equal:

$$a(\alpha) = \frac{1}{2}f_{xx}(0,\mu),\ \ b(\alpha) = \frac{1}{6}f_{xxx}(0,\mu).$$

Define a smooth function $\sigma = \sigma(\alpha)$ and make a change of coordinate

$$x = y + \sigma y^2.$$

This transformation is invertible in some neighborhood of the origin, and its inverse can be found by the method of unknown coefficients:

$$y = x - \sigma x^2 + 2\sigma^2 x^3 + O(x^4).$$

Using the previous transformation and its inverse, we get

$$\tilde{y} = \mu y + (a + \sigma\mu - \sigma\mu^2)y^2 + (b + 2\sigma a - 2\sigma\mu(\sigma\mu + a) + 2\sigma^2\mu^3)y^3 + O(y^4).$$

Setting

$$\delta(\alpha) = \frac{a(\alpha)}{\mu^2(\alpha) - \mu(\alpha)}.$$

Since $\mu^2(0) - \mu(0) = 2 \neq 0$, the quadratic term can be "killed" for all sufficiently small $|\alpha|$. And we get

$$\tilde{y} = \mu y + (b + \frac{2a^2}{\mu^2 - \mu})y^3 + O(y^4) = -(1+\alpha)y + c(\alpha)y^3 + O(y^4),$$

where $c(\alpha)$ is a smooth function such that

$$c(0) = a^2(0) + b(0) = \frac{1}{4}(f_{xx}(0,0))^2 + \frac{1}{6}f_{xxx}(0,0).$$

So $c(0) \neq 0$ since $\frac{1}{4}(f_{xx}(0,0))^2 + \frac{1}{6}f_{xxx}(0,0) \neq 0$. Take

$$y = \frac{\xi}{\sqrt{|c(\alpha)|}}.$$

The system takes the desired form:

$$\tilde{\xi} = -(1+\alpha)\xi + s\xi^3 + O(\xi^4).$$

Where $s = sign\ c(0) = \pm 1$.

■

Lemma 2.7. *[5] The map*

$$x \mapsto -(1+\mu)x + x^3 + O(x^4)$$

is locally topologically equivalent near the origin to the map

$$x \mapsto -(1+\mu)x + x^3.$$

We arrive to the following general result:

Theorem 2.8 (Topological normal form for the flip bifurcation)**.** *[5]*
Any generic, one-parameter system

$$x \mapsto f(x, \mu)$$

having at $\mu = 0$ the fixed point $x_0 = 0$ with $\lambda = f_x(0,0) = -1$, is locally topologically equivalent near the origin to one of the following normal forms:

$$\xi \mapsto -(1+\alpha)\xi + s\xi^3.$$

Consider the flip bifurcation case for any n-dimensional map

$$\tilde{x} = Ax + G(x),\ x \in \mathbb{R} \tag{2.5.5}$$

where $G(x) = O(\|x\|^2)$ is a smooth function and its Taylor expansion is

$$G(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\|x^4\|)$$

where

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j)$$

and

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j).$$

And the Jacobian matrix A has the eigenvalue $\lambda = -1$ and the corresponding critical eigenspace T^c is one-dimensional and spanned by an eigenvector $\hat{q} \in \mathbb{R}^n$ such that $A\hat{q} = \lambda\hat{q}$. Let $\hat{p} \in \mathbb{R}^n$ be the adjoint eigenvector, that is, $A^T\hat{p} = \lambda\hat{p}$, where A^T is the transposed matrix. Normalize $\hat{p}$ with respect to $\hat{q}$ such that $\langle \hat{p}, \hat{q} \rangle = 1$. Let T^{su} denote an $(n-1)$-dimensional linear eigenspace of A corresponding to all eigenvalues other than λ. Note that the matrix $(A - \lambda I_n)$ has common invariant spaces with the matrix A, so we conclude that $y \in T^{su}$ if and only if $\langle \hat{p}, y \rangle = 0$.
Any vector $x \in \mathbb{R}^n$ can be decomposed as

$$x = u\hat{q} + y$$

where $u\hat{q} \in T^c$, $y \in T^{su}$ and

$$u = \langle \hat{p}, x \rangle.$$

$$y = x - \langle \hat{p}, x \rangle \hat{q}.$$

In the coordinates (u, y), the map (2.5.5) can be written as

$$\tilde{u} = \lambda u + \langle \hat{p}, G(u\hat{q} + y) \rangle,$$

$$\tilde{y} = Ay + G(u\hat{q} + y) - \langle \hat{p}, G(u\hat{q} + y) \rangle \hat{q}.$$

Using Taylor expansions, the last two equations can be written as

$$\begin{aligned} \tilde{u} &= \lambda u + \frac{1}{2}\delta u^2 + u\langle b, y \rangle + \frac{1}{6}\sigma u^3 + ... \\ \tilde{y} &= Ay + \frac{1}{2}au^2 + ... \end{aligned} \tag{2.5.6}$$

where $u = \mathbb{R}$, $y \in \mathbb{R}^n$, δ, $\sigma \in \mathbb{R}$, a, $b \in \mathbb{R}^n$ and $\langle b, y \rangle = \sum_{i=1}^{n} b_i y_i$ is the standard scalar product, and can be expressed as

$$\langle b, y \rangle = \langle b, B(\hat{q}, y) \rangle.$$

The center manifold of (2.5.6) has the representation

$$y = V(u) = \frac{1}{2}w_2u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset \mathbb{R}^n$, so that $\langle \hat{p}, w_2 \rangle = 0$. The vector w_2 satisfies

$$(A - I_n)w_2 + a = 0.$$

We have $\lambda = 1$ is not an eigenvalue of A, so the matrix $(A - I_n)$ is invertible in $\mathbb{R}^n$. Thus, we have

$$w_2 = -(A - I_n)^{-1}a$$

and the restriction of (2.5.6) to the center manifold takes the form

$$\tilde{u} = -u + \frac{1}{2}\delta u^2 + \frac{1}{6}(\sigma - 3\langle \hat{q}, (A - I_n)^{-1}a \rangle)u^3 + O(u^4)$$

where $\delta = \langle \hat{p}, B(\hat{q}, \hat{q}) \rangle$, $\sigma = \langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q}) \rangle$ and $a = B(\hat{q}, \hat{q}) - \langle \hat{p}, B(\hat{q}, \hat{q}) \rangle \hat{q}$.

Using the identity

$$(A - I_n)\hat{q} = \frac{1}{2}\hat{q},$$

the restricted map can be written as

$$\tilde{u} = -u + a(0)u^2 + b(0)u^3 + O(u^4) \tag{2.5.7}$$

where

$$a(0) = \frac{1}{2}\langle \hat{p}, B(\hat{q}, \hat{q}) \rangle,$$

and

$$b(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q}) \rangle - \frac{1}{4}(\langle \hat{p}, B(\hat{q}, \hat{q}) \rangle)^2 - \frac{1}{2}\langle \hat{p}, B(\hat{q}, (A - I_n)^{-1}B(\hat{q}, \hat{q})) \rangle.$$

The map (2.5.7) can be transformed to the normal form

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4)$$

where

$$c(0) = a^2(0) - b(0).$$

Thus,the critical normal form coefficient $c(0)$, allows us to predict the direction of bifurcation of the period-two cycle. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q})\rangle - \frac{1}{2}\langle \hat{p}, B(\hat{q}, (A - I_n)^{-1}B(\hat{q}, \hat{q}))\rangle.$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point. [5]

2.6 Bifurcation of two-dimensional maps

In two-dimensional maps case we have a new bifurcation, the Neimark-Sacker bifurcation. In addition to the previous types of bifurcation. This section will contain details about Neimark-Sacker bifurcation.

For two-dimensional maps non-hyperbolic fixed points are those where the Jacobian matrix has eigenvalue on the unit circle.

Let

$$f(x, \mu),\ x \in \mathbb{R}^2,\ \mu \in \mathbb{R}$$

be a C^r one-parameter family of two-dimensional maps, where $r > 6$. with fixed point $(\bar{x}, \mu^*)$. If $(\bar{x}, \mu^*) \neq (0, 0)$ we transform this fixed point to the origin. Let $A = Jf(0, 0)$ be the Jacobian matrix of $f(x, \mu)$ with $\rho(A) = 1$. Then we consider the following three cases:

1. If A has one real value equal to 1, then we have one of the following types of bifurcation (saddle-node bifurcation, pitchfork bifurcation, or transcritical bifurcation).
2. If A has one real value equal to -1, then we have a period-doubling bifurcation.
3. If A has a pair of complex conjugate eigenvalues of modulus 1, then we have the Neimark-Sacker bifurcation.

[10]

We saw in chapter 1 that the stability region in the trace-determinant plane is enclosed the lines $det(A) = tr(A) - 1$, $det(A) = -tr(A) - 1$, and $det(A) = 1$, where A is the Jacobian matrix at the fixed point. The following theorem shows the importance of those lines in studying the bifurcation of two-dimensional maps.

Theorem 2.9. *[10] Consider the two-dimensional map*

$$x \mapsto f(x,\mu), \;\; x \in \mathbb{R}, \;\; \mu \in \mathbb{R}.$$

Let $(\bar{x}, \mu^*)$ be a fixed point of $f(x,\mu)$ and $A = Jf(\bar{x},\mu^*)$. Then the following statements hold:

1. *If $det(A) = tr(A) - 1$, then the eigenvalues of A are $\lambda_1 = det(A)$ and $\lambda_2 = 1$.*
2. *If $det(A) = -tr(A) - 1$, then the eigenvalues of A are $\lambda_1 = -det(A)$ and $\lambda_2 = -1$.*
3. *If $|tr(A)| - 1 < det(A)$ and $det(A) = 1$, then A has a pair of complex conjugate eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$ where $\theta = \cos^{-1}(\frac{tr(A)}{2})$.*

Proof: Consider the map $f(x,\mu)$ with Jacobian matrix $A = Jf(\bar{x},\mu^*)$.
We have

$$\lambda_{1,2} = \frac{tr(A) \pm \sqrt{(tr(A)^2 - 4det(A)}}{2}$$

1. Let $det(A) = tr(A) - 1$. Then

$$tr(A)^2 - 4det(A) = (det(A) - 1)^2 > 0.$$

   So

$$\lambda_{1,2} = \frac{det(A) + 1 \pm \sqrt{(det(A)-1)^2}}{2} = \frac{det(A) + 1 \pm (det(A) - 1)}{2}.$$

   This implies that

$$\lambda_1 = det(A), \;\; \lambda_2 = 1.$$

2. Let $det(A) = -tr(A) - 1$. Then

$$tr(A)^2 - 4det(A) = (det(A) - 1)^2 > 0.$$

   So

$$\lambda_{1,2} = \frac{-det(A) - 1 \pm \sqrt{(det(A)-1)^2}}{2} = \frac{-det(A) - 1 \pm (det(A) - 1)}{2}.$$

This implies that

$$\lambda_1 = -det(A), \ \lambda_2 = -1.$$

3. Let $|tr(A)| - 1 < det(A)$ and $det(A) = 1$. Then

$$tr(A)^2 - 4det(A) = tr(A)^2 - 4 < (det(A) + 1)^2 - 4 = 0.$$

This implies that A has a pair of complex conjugate eigenvalues

$$\lambda_{1,2} = \frac{tr(A) \pm \sqrt{4det(A) - (tr(A))^2}}{2}.$$

Since $det(A) = 1$ we have

$$\lambda_{1,2} = \frac{tr(A) \pm \sqrt{4 - (tr(A))^2}}{2}.$$

Thus, $\lambda_{1,2} = re^{\pm i\theta}$ where $r = |\lambda_{1,2}| = 1$ and $\theta = \tan^{-1}\left(\frac{\sqrt{det(A) - (\frac{tr(A)}{2})^2}}{\frac{tr(A)}{2}}\right) = \cos^{-1}\left(\frac{tr(A)}{2}\right)$.

■

Corollary 2.9.1. *[10] Let*

$$x \mapsto f(x, \mu), \ x \in \mathbb{R}^2, \ \mu \in \mathbb{R} \tag{2.6.1}$$

be a one-parameter family of two-dimensional maps, with fixed point $(\bar{x}, \mu^*)$ and $A = Jf(\bar{x}, \mu^*)$. Then the following statements hold:

1. *If $det(A) = tr(A) - 1$, then the system (2.6.1) undergoes a saddle-node bifurcation.*

2. *If $det(A) = -tr(A) - 1$, then the system (2.6.1) undergoes a period-doubling bifurcation.*

3. *If $|tr(A)| - 1 < det(A)$ and $det(A) = 1$, then the system (2.6.1) undergoes a Neimark-Sacker bifurcation.*

2.7 The Neimark-Sacker bifurcation

The Neimark-Sacker bifurcation exists when we have a pair of complex conjugate eigenvalues of modulus 1.
Any map undergoes a Neimark-Sacker bifurcation has a unique closed invariant curve bifurcates from the fixed point as the bifurcation parameter passes through zero. The closed invariant curve can be stable or unstable as the bifurcation is supercritical or subcritical, respectively.

2.7.1 The normal form of the Neimark-Sacker bifurcation

Consider the following two-dimensional discrete-time system depending on one parameter:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (1+\mu) \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + ({x_1}^2 + {x_2}^2) \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.7.1}$$

where μ is the parameter, $\theta = \theta(\mu)$, $a = a(\mu)$, and $b = b(\mu)$ are smooth functions, and $0 < \theta(0) < \pi$, $a(0) \neq 0$.
This system has the fixed point $x_1 = x_2 = 0$ for all μ with Jacobian matrix

$$A = (1+\mu) \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

The matrix has eigenvalues $\lambda_{1,2} = (1+\mu)e^{\pm\theta}$, which makes the map (2.7.1) invertible near the origin for all small $|\mu|$. We can see that the fixed point at the origin is non-hyperbolic at $\mu = 0$.
To analyze the corresponding bifurcation, introduce the complex variable

$$z = x_1 + ix_2,\ \bar{z} = x_1 - ix_2,\ |z|^2 = {x_2}^2 + {x_2}^2,$$

and take $d = a + ib$. The equation for z is

$$z \mapsto e^{i\theta} z(1 + \mu + d|z|^2) = \alpha z + cz|z|^2$$

where $\alpha = \alpha(\mu) = (1+\mu)e^{i\theta(\mu)}$ and $c = c(\mu) = e^{i\theta(\mu)}d(\mu)$ are complex functions of the parameter μ.
Using the representation $z = \rho e^{i\varphi}$, we obtain for $\rho = |z|$

$$\rho \mapsto \rho|1+\mu+d(\mu)\rho^2|.$$

But

$$|1+\mu+d(\mu)\rho^2| = (1+\mu)\sqrt{1+\frac{2a(\mu)}{1+\mu}\rho^2+\frac{|d(\mu)|^2}{(1+\mu)^2}\rho^4} = 1+\mu+a(\mu)\rho^2+O(\rho^3).$$

So we get the following polar form for (2.7.1)

$$\begin{cases} \rho \mapsto \rho(1+\mu+a(\mu)\rho^2)+\rho^4 R_\mu(\rho), \\ \varphi \mapsto \varphi+\theta(\mu)+\rho^2 Q_\mu(\rho). \end{cases}$$

Where R and Q are smooth functions of (ρ,μ). Bifurcations of the systems' phase portrait as μ passes through zero can easily be analyzed using the latter form, since the mapping for ρ is independent of φ.
The first equation

$$\varphi \mapsto \varphi+\theta(\mu)+\rho^2 Q_\mu(\rho)$$

defines a one-dimensional dynamical system that has the fixed point $\rho = 0$ for all values of μ. The point is linearly stable if $\mu < 0$; for $\mu > 0$ the point becomes linearly unstable. The stability of the fixed point at $\mu = 0$ is determined by the sign of the coefficient $a(0)$. Suppose that $a(0) < 0$; then the origin is (nonlinearly) stable at $\mu = 0$. Moreover, the previous ρ-map has an additional stable fixed point

$$\rho_0 = \sqrt{-\frac{\mu}{a(\mu)}}+O(\mu)$$

for $\mu > 0$.
The φ-map describes a rotation by an angle depending on ρ and μ; it is approximately equal to $\theta(\mu)$. Thus, we obtain the bifurcation diagram for the original two-dimensional system (2.7.1) by superposition of its polar form.

The system always has a fixed point at the origin. This point is stable for $\mu < 0$ and unstable for $\mu > 0$. The invariant curves of the system near the origin look like

the orbits near the stable focus of a continuous-time system for $\mu < 0$ and like orbits near the unstable focus for $\mu > 0$. At the critical parameter value $\mu = 0$ the point is nonlinearly stable. The fixed point is surrounded for $\mu > 0$ by an isolated closed invariant curve that is unique and stable. The curve is a circle of radius $\rho_0(\mu)$. All orbits starting outside or inside the closed invariant curve, except at the origin, tend to the curve under iterations of (2.7.1). This is a Neimark-Sacker bifurcation.
This bifurcation can also be presented in (x_1, x_2, μ)-space. The appearing family of closed invariant curves, parametrized by μ, forms a paraboloid surface. The case $a(0) > 0$ can be analyzed in the same way. The system undergoes the Neimark-Sacker bifurcation at $\mu = 0$ but there is an unstable closed invariant curve that disappears when μ crosses zero from negative to positive values.[5]

2.7.2 Generic Neimark-Sacker bifurcation

We shall now prove that any generic two-dimensional system undergoing a Neimark-Sacker bifurcation can be transformed into the form (2.7.1). Consider a system

$$x \mapsto f(x, \mu),\ \ x = (x_1, x_2)^T \in \mathbb{R}^2,\ \ \mu \in \mathbb{R}$$

with a smooth function f, which has at $\mu = 0$ the fixed point $x = 0$ with simple eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$. By the Implicit Function Theorem, the system has a unique fixed point $x_0(\mu)$ in some neighborhood of the origin for all sufficiently small $|\mu|$, since $\lambda = 1$ is not an eigenvalue of the Jacobian matrix. We can perform a parameter-dependent coordinate shift, placing this fixed point at the origin. Therefore, we may assume without loss of generality that $x = 0$ is the fixed point of the system for $|\mu|$ sufficiently small. Thus, the system can be written as

$$x \mapsto A(\mu)x + F(x, \mu) \tag{2.7.2}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms, $F(x, \mu) = 0$ for all sufficiently small $|\mu|$. The Jacobian matrix $A(\mu)$ has two multipliers

$$\lambda_{1,2} = r(\mu)e^{\pm i\varphi(\mu)}$$

where $r(0) = 1$, $\varphi(0) = \theta_0$. Thus, $r(\mu) = 1 + \beta(\mu)$ for some smooth function $\beta(\mu)$ with $\beta(0) = 0$. Suppose that $\beta'(0) \neq 0$. Then, we can use β as a new parameter and express the multipliers in terms of β: $\lambda_1(\beta) = \lambda(\mu)$, $\lambda_2(\beta) = \bar{\lambda}(\mu)$, where

$$\lambda(\beta) = (1+\beta)e^{i\theta(\beta)}$$

with a smooth function $\theta(\beta)$ such that $\theta(0) = \theta_0$.

Lemma 2.10. *[5] By the introduction of a complex variable and a new parameter, system (2.7.2) can be transformed for all sufficiently small $|\mu|$ into the following form:*

$$z \mapsto \lambda(\beta)z + g(z, \bar{z}, \beta), \tag{2.7.3}$$

where $\beta \in \mathbb{R}$, $z \in \mathbb{C}$, $\lambda(\beta) = (1+\beta)e^{i\theta(\beta)}$, and g is a complex-valued smooth function of z, $\bar{z}$, and β whose Taylor expansion with respect to $(z, \bar{z})$ contains quadratic and higher-order terms:

$$g(z, \bar{z}, \beta) = \sum_{k+1\geq 2} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l,$$

with $l,\ k = 0,\ 1,\ 2,\ \ldots.$

Lemma 2.11. *[5] The map*

$$z \mapsto \lambda z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + O(|z|^3) \tag{2.7.4}$$

where $\lambda = \lambda(\beta) = (1+\beta)e^{i\theta(\beta)}$, $g_{ij} = g_{ij}(\beta)$ can be transformed by an invertible parameter-dependent change of complex coordinate

$$z = v + \frac{h_{20}}{2}v^2 + h_{11}v\bar{v} + \frac{h_{02}}{2}\bar{v}^2,$$

for all sufficiently small $|\beta|$, into a map without quadratic terms:

$$v \mapsto \lambda v + O(|v|^3),$$

provided that

$$e^{i\theta_0} \neq 1,\ \ e^{3i\theta_0} \neq 1.$$

Proof: The inverse change of variables is given by

$$v = z - \frac{h_{20}}{2}z^2 - h_{11}z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + O(|z|^3).$$

Therefore, in the new coordinate v, the map (2.7.4) takes the form

$$\tilde{v} = \lambda v + \frac{1}{2}(g_{20} + (\lambda - \lambda^2)h_{20})v^2 + (g_{11} + (\lambda - |\lambda|^2)h_{11})v\bar{v} + \frac{1}{2}(g_{02} + (\lambda - \lambda^2)h_{02})\bar{v}^2 + O(|v|^3).$$

Thus, taking

$$h_{20} = \frac{g_{20}}{\lambda - \lambda^2},\ h_{11} = \frac{g_{11}}{|\lambda|^2 - \lambda},\ h_{11} = \frac{g_{02}}{\bar{\lambda}^2 - \lambda},$$

we "kill" all the quadratic terms in (2.7.4). These substitutions are valid if the denominators are nonzero for all sufficiently small $|\beta|$ including $\beta = 0$. Indeed, this is the case, since

$$\lambda^2(0) - \lambda(0) = e^{i\theta_0}(e^{i\theta_0} - 1) \neq 0,$$

$$|\lambda(0)|^2 - \lambda(0) = 1 - e^{i\theta_0} \neq 0,$$

$$\bar{\lambda}^2(0) - \lambda(0) = e^{i\theta_0}(e^{-3i\theta_0} - 1) \neq 0,$$

due to our restrictions on θ_0. ■

Assuming that we have removed all quadratic terms, let us try to remove the cubic terms.

Lemma 2.12. *[5] The map*

$$z \mapsto \lambda z + \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}z\bar{z}^2 + \frac{g_{03}}{6}\bar{z}^3 + O(|z|^4)$$

where $\lambda = \lambda(\beta) = (1+\beta)e^{i\theta(\beta)}$, $g_{ij} = g_{ij}(\beta)$ can be transformed by an invertible parameter-dependent change of coordinates

$$z = v + \frac{h_{30}}{6}v^3 + \frac{h_{21}}{2}v^2\bar{v} + \frac{h_{12}}{2}v\bar{v}^2 + \frac{h_{03}}{6}\bar{v}^3,$$

for all sufficiently small $|\beta|$, into a map without quadratic terms:

$$v \mapsto \lambda v + c_1 v^2\bar{v} + O(|v|^4),$$

provided that

$$e^{2i\theta_0} \neq 1,\ e^{4i\theta_0} \neq 1.$$

Proof: The inverse transformation is

$$v = z - \frac{h_{30}}{6}z^3 - \frac{h_{21}}{2}z^2\bar{z} - h_{12}z\bar{z}^2 - \frac{h_{03}}{6}\bar{z}^3 + O(|z|^4).$$

Therefore

$$\tilde{v} = \lambda v + \frac{1}{6}(g_{30} + (\lambda - \lambda^3)h_{30})v^3 + \frac{1}{2}(g_{21} + (\lambda - \lambda|\lambda|^2)h_{21})v^2\bar{v} + \frac{1}{2}(g_{12} + (\lambda - \bar{\lambda}|\lambda|^2)h_{12})v\bar{v}^2 + \frac{1}{6}(g_{03} + (\lambda - \bar{\lambda}^3)h_{03})\bar{v}^3 + O(|v|^4).$$

Thus, taking

$$h_{30} = \frac{g_{30}}{\lambda^3 - \lambda}, \; h_{12} = \frac{g_{12}}{\bar{\lambda}|\lambda|^2 - \lambda}, \; h_{03} = \frac{g_{03}}{\bar{\lambda}^3 - \lambda},$$

we can annihilate all cubic terms in the resulting map except the $v^2\bar{v}$-term, which must be treated separately. The substitutions are valid since all the involved denominators are nonzero for all sufficiently small $|\beta|$ due to the assumptions concerning θ_0.

One can also try to eliminate the $v^2\bar{v}$-term by formally setting

$$h_{21} = \frac{g_{21}}{\lambda(1 - |\lambda|^2)}.$$

This is possible for small $\beta \neq 0$, but the denominator vanishes at $\beta = 0$ for all θ_0. Thus, no extra conditions on θ_0 would help. To obtain a transformation that is smoothly dependent on β, set $h_{21} = 0$, that results in

$$c_1 = \frac{g_{21}}{2}.$$

■

Lemma 2.13 (Normal form for the Neimark-Sacker bifurcation)**.** *The map*

$$z \mapsto \lambda z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}z\bar{z}^2 + \frac{g_{03}}{6}\bar{z}^3 + O(|z|^4)$$

where $\lambda = \lambda(\beta) = (1+\beta)e^{i\theta(\beta)}$*,* $g_{ij} = g_{ij}(\beta)$*, and* $\theta_0 = \theta(0)$ *is such that* $e^{ik\theta_0} \neq 1$ *for* $k = 1,\ 2,\ 3,\ 4$*, can be transformed by an invertible parameter-dependent change of complex coordinates, which is smoothly dependent on the parameter,*

$$z = v + \frac{h_{20}}{2}v^2 + h_{11}v\bar{v} + \frac{h_{02}}{2}\bar{v}^2 + \frac{h_{30}}{6}v^3 + \frac{h_{12}}{2}v\bar{v}^2 + \frac{h_{03}}{6}\bar{v}^3,$$

for all sufficiently small $|\beta|$, into a map with only the resonant cubic term:

$$v \mapsto \lambda v + c_1 v^2\bar{v} + O(|v|^4),$$

where $c_1 = c_1(\beta)$.

The truncated superposition of the transformations defined in the two previous lemmas gives the required coordinate change. First, annihilate all the quadratic terms. This will also change the coefficients of the cubic terms. The coefficient of $v^2\bar{v}$ will be $\frac{1}{2}\tilde{g}_{21}$, instead of $\frac{1}{2}g_{21}$. Then, eliminate all the cubic terms except the resonant one. The coefficient of this term remains $\frac{1}{2}g_{21}$. Thus, all we need to compute to get the coefficient of c_1 in terms of the given equation is a new coefficient $\frac{1}{2}\tilde{g}_{21}$ of the $v^2\bar{v}$-term after the quadratic transformation. The computations result in the following expression for $c_1(\beta)$:

$$c_1 = \frac{g_{20}g_{11}(\bar{\lambda} - 3 - 2\lambda)}{2(\lambda^2 - \lambda)(\bar{\lambda} - 1)} + \frac{|g_{11}|^2}{1 - \bar{\lambda}} + \frac{|g_{02}|^2}{2(\lambda^2 - \bar{\lambda})} + \frac{g_{21}}{2},$$

which gives, for the critical value of c_1,

$$c_1(0) = \frac{g_{20}(0)g_{11}(0)(\bar{\lambda_0} - 3 - 2\lambda_0)}{2({\lambda^2}_0 - \lambda_0)(\bar{\lambda}_0 - 1)} + \frac{|g_{11}(0)|^2}{1 - \bar{\lambda}_0} + \frac{|g_{02}(0)|^2}{2({\lambda^2}_0 - \bar{\lambda}_0)} + \frac{g_{21}(0)}{2}, \qquad (2.7.5)$$

where $\lambda_0 = e^{i\theta_0}$.

We now summarize the obtained results in the following theorem.

Theorem 2.14. *[5] Suppose a two-dimensional discrete-time system*

$$x \mapsto f(x, \mu),\ \ x \in \mathbb{R}^2,\ \ \mu \in \mathbb{R}$$

with smooth function f which has, for all sufficiently small $|\mu|$, $x = 0$ as a fixed point with eigenvalues

$$\lambda_{1,2}(\mu) = r(\mu)e^{\pm i\varphi(\mu)}$$

where $r(0) = 1$, $\varphi(0) = \theta_0$.
Let the following conditions be satisfied:

1. $r'(0) \neq 0$;
2. $e^{ik\theta_0} \neq 1$ *for* $k = 1,\ 2,\ 3,\ 4$.

Then, there are smooth invertible coordinate and parameter changes transforming the system into

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto (1+\beta)\begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + (y_1^2+y_2^2)\begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix}\begin{pmatrix} a(\beta) & -b(\beta) \\ b(\beta) & a(\beta) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\|y\|^4) \tag{2.7.6}$$

with $\theta(0) = \theta_0$ *and* $a(0) = Re(e^{-i\theta_0}c_1(0))$, *where* $c_1(0)$ *is given by the formula (2.7.5).*

Proof: The only thing left to verify is the formula for $a(0)$. Indeed, by Lemmas 2.10, 2.11, and 2.12, the system can be transformed to the complex *Poincare′* normal form,

$$v \mapsto \lambda(\beta)v + c_1(\beta)v|v|^2 + O(|v|^4),$$

for $\lambda(\beta) = (1+\beta)e^{i\theta(\beta)}$. This map can be written as

$$v \mapsto e^{i\theta(\beta)}(1+\beta+d(\beta)|v|^2)v + O(|v|^4),$$

where $d(\beta) = a(\beta) + ib(\beta)$ for some real functions $a(\beta),\ b(\beta)$. A return to real coordinates (y_1, y_2), $v = y_1 + iy_2$, gives system (2.7.6). Finally,

$$a(\beta) = Re(e^{-i\theta(\beta)}c_1(\beta).$$

Thus

$$a(0) = Re(e^{-i\theta_0}c(0).$$

■

Theorem 2.15 (Generic Neimark-Sacker bifurcation). *[5] For any generic two-dimensional one-parameter system*

$$x \mapsto f(x, \mu)$$

having at $\mu = 0$ the fixed point $x_0 = 0$ with complex eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$ there is a neighborhood of x_0 in which a unique closed invariant curve bifurcates from x_0 as μ passes through zero.

Consider the map

$$\tilde{x} = Ax + G(x), \; x \in \mathbb{R} \tag{2.7.7}$$

where the Jacobian matrix A has a simple pair of complex eigenvalues of modulus one, $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$ and these are only eigenvalues with $|\lambda| = 1$ and $G(x) = O(\|x\|^2)$ is a smooth function and its Taylor expansion is

$$G(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\|x^4\|)$$

where

$$B_i(x, y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j)$$

and

$$C_i(x, y, z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j).$$

Let $\hat{q} \in \mathbb{C}^n$ be a complex eigenvector corresponding to λ_1:

$$A\hat{q} = e^{i\theta_0}\hat{q}, \; A\bar{\hat{q}} = e^{-i\theta_0}\bar{\hat{q}}.$$

Introduce also the adjoint eigenvector $\hat{p} \in \mathbb{C}^n$ having the properties

$$A^T\hat{p} = e^{-i\theta_0}\hat{p}, \; A^T\bar{\hat{p}} = e^{i\theta_0}\bar{\hat{p}},$$

and satisfying the normalization

$$\langle \hat{p}, \hat{q} \rangle = 1$$

where $\langle \hat{p}, \hat{q} \rangle = \sum_{i=1}^{n} \bar{\hat{p}}_i \hat{q}_i$ is the standard scalar product in $\mathbb{C}^n$. The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and spanned by $\{Re(\hat{q}), Im(\hat{q})\}$.

The real space T^s corresponding to real eigenvalues of A is $(n-2)$-dimensional. $y \in T^{su}$ if and only if $\langle \hat{p}, y\rangle = 0$. Note that $y \in \mathbb{R}^n$ is real, while $\hat{p} \in \mathbb{C}^n$.
Any vector $x \in \mathbb{R}^n$ can be decomposed as

$$x = z\hat{q} + \bar{z}\bar{\hat{q}} + y$$

where $z \in \mathbb{C}^1$, $z\hat{q} + \bar{z}\bar{\hat{q}} \in T^c$ and $y \in T^s$. The complex variable z is a coordinate on T^c. We have

$$\begin{aligned} z &= \langle \hat{p}, x\rangle \\ y &= x - \langle \hat{p}, x\rangle \hat{q} - \langle \bar{\hat{p}}, x\rangle \bar{\hat{q}} \end{aligned}$$

In these coordinates, the map (2.7.7) takes the form

$$\begin{aligned} \tilde{z} &= e^{i\theta_0} z + \langle \hat{p}, G(z\hat{q} + \bar{z}\bar{\hat{q}} + y)\rangle \\ \tilde{y} &= Ay + G(z\hat{q} + \bar{z}\bar{\hat{q}} + y) - \langle \hat{p}, G(z\hat{q} + \bar{z}\bar{\hat{q}} + y)\rangle \hat{q} - \langle \bar{\hat{p}}, G(z\hat{q} + \bar{z}\bar{\hat{q}} + y)\rangle \bar{\hat{q}}. \end{aligned}$$

The previous system can be written as

$$\begin{aligned} \tilde{z} &= e^{i\theta_0} z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} + \langle G_{10}, y\rangle z + \langle G_{10}, y\rangle \bar{z} \\ \tilde{y} &= Ay + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \frac{1}{2}H_{21}z^2\bar{z}, \end{aligned}$$

where $G_{20},\ G_{11},\ G_{02},\ G_{21} \in \mathbb{C}^1$ and $G_{01},\ G_{10},\ H_{ij} \in \mathbb{C}^n$ and the scaler product is in $\mathbb{C}^n$.

The complex numbers and vectors can be computed by

$$G_{20} = \langle \hat{p}, B(\hat{q}, \hat{q})\rangle,\ G_{11} = \langle \hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle,\ G_{02} = \langle \hat{p}, B(\bar{\hat{q}}, \bar{\hat{q}})\rangle,\ \ G_{21} = \langle \hat{p}, C(\hat{q}, \hat{q}, \bar{\hat{q}})\rangle$$

and

$$\begin{aligned} H_{20} &= B(\hat{q}, \hat{q}) - \langle \hat{p}, B(\hat{q}, \hat{q})\rangle \hat{q} - \langle \bar{\hat{p}}, B(\hat{q}, \hat{q})\rangle \bar{\hat{q}}, \\ H_{11} &= B(\hat{q}, \bar{\hat{q}}) - \langle \hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle \hat{q} - \langle \bar{\hat{p}}, B(\hat{q}, \bar{\hat{q}})\rangle \bar{\hat{q}}, \end{aligned}$$

and

$$\langle G_{10}, y\rangle = \langle \hat{p}, B(\hat{q}, y)\rangle,\ \langle G_{01}, y\rangle = \langle \hat{p}, B(\bar{\hat{q}}, y)\rangle.$$

The center manifold in the previous system has the representation

$$Y = V(z, \bar{z}) = \frac{1}{2}v_{20}z^2 + v_{11}z\bar{z} + \frac{1}{2}v_{02}\bar{z}^2$$

where $\langle \hat{q}, v_{ij} \rangle = 0$. The vectors $v_{ij} \in \mathbb{C}^n$ can be found from linear equations

$$(e^{2i\theta_0}I - A)v_{20} = H_{20},$$

$$(I - A)v_{11} = H_{11},$$

$$(e^{-2i\theta_0}I - A)v_{02} = H_{02}.$$

These equations have unique solutions. The matrix $(I - A)$ is invertible because 1 is not an eigenvalue of A. If

$$e^{3i\theta_0} \neq 1$$

the matrices $(e^{\pm 2i\theta_0}I - A)$ are also invertible in $\mathbb{C}^n$ because $e^{\pm 2i\theta_0}$ are not eigenvalues of A. Thus, generically the restricted map can be written as

$$\tilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}[G_{21} + 2\langle \bar{\hat{p}}, B(\hat{q}, (I - A)^{-1}H_{11})\rangle +$$
$$\langle \bar{\hat{p}}, B(\bar{\hat{q}}, (e^{2i\theta_0}I - A)^{-1}H_{20})\rangle]z^2\bar{z} + ...$$

taking into account the identities

$$(I - A)^{-1}\hat{q} = \frac{1}{1 - e^{i\theta_0}}\hat{q},\ (e^{2i\theta_0}I - A)^{-1}\hat{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0} - 1}\hat{q},\ (I - A)^{-1}\bar{\hat{q}} = \frac{1}{1 - e^{i\theta_0}}\bar{\hat{q}},$$

and

$$(e^{2i\theta_0}I - A)^{-1}\bar{\hat{q}} = \frac{e^{-i\theta_0}}{e^{i\theta_0} - 1}\bar{\hat{q}}.$$

Also z can be written using the map

$$\tilde{z} = e^{i\theta_0}z + \sum_{k,l \geq 2} \frac{1}{k!j!}g_{kj}z^k\bar{z}^j, \tag{2.7.8}$$

where $g_{20} = \langle \hat{p}, B(\hat{q}, \hat{q})\rangle$, $g_{11} = \langle \hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle$, $g_{02} = \langle \hat{p}, B(\bar{\hat{q}}, \bar{\hat{q}})\rangle$, and

$$g_{21} = \langle \hat{p}, C(\hat{q}, \hat{q}, \bar{\hat{q}})\rangle + 2\langle \hat{p}, B(\hat{q}, (I - A)^{-1}B(\hat{q}, \bar{\hat{q}}))\rangle +$$
$$\langle \hat{p}, B(\bar{\hat{q}}, (e^{2i\theta_0}I - A)^{-1}B(\hat{q}, \hat{q}))\rangle + \frac{e^{-i\theta_0}(1 - 2e^{i\theta_0})}{1 - e^{-i\theta_0}}\langle \hat{p}, B(\hat{q}, \hat{q})\rangle\langle \hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle$$
$$- \frac{2}{1 - e^{-i\theta_0}}|\langle \hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle|^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0} - 1}|\langle \hat{p}, B(\bar{\hat{q}}, \bar{\hat{q}})\rangle|^2.$$

As $e^{ik\theta_0} \neq 1$, the map (2.7.8) can be transformed into the form

$$\bar{z} = e^{i\theta_0} z(1 + d(0))|z^4|$$

where $a(0) = Re\{d(0)\}$, that determines the direction of bifurcation of a closed invariant curve, can be computed by formula

$$a(0) = Re\left(\frac{e^{-i\theta_0} g_{21}}{2}\right) - Re\left(\frac{(1 - e^{2i\theta_0})e^{-2i\theta_0}}{2(1 - e^{i\theta_0})} g_{20} g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2.$$

This compact formula allows us to verify the non-degeneracy of the nonlinear terms at a non-resonant Neimark-Sacker bifurcation of n-dimensional maps with $n \geq 2$.[5]

3. DYNAMICS OF $X_{N+1} = \frac{\alpha+\beta X_{N-1}}{A+BX_N^2+CX_{N-1}}$

In this chapter we consider the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n^2 x_n^2 + Cx_{n-1}}, \; n = 0, \; 1, \; 2, \; ... \tag{3.0.1}$$

with positive parameters $\alpha, \; \beta, \; A, \; B, \; C$, and non-negative initial conditions.

We will focus on the dynamic behavior of the positive fixed point and the type of bifurcation exist where the change of stability occurs. Then we introduce some Matlab codes that use our results.

3.1 Change of variables

Consider Equation (3.0.1), let

$$x_n = \frac{\sqrt{A}}{\sqrt{B}} y_n.$$

Then

$$x_{n+1} = \frac{\sqrt{A}}{\sqrt{B}} y_{n+1},$$

and

$$x_{n-1} = \frac{\sqrt{A}}{\sqrt{B}} y_{n-1}.$$

So Equation (3.0.1) becomes

$$\frac{\sqrt{A}}{\sqrt{B}} y_{n+1} = \frac{\alpha + \beta \frac{\sqrt{A}}{\sqrt{B}} y_{n-1}}{A + B\frac{A}{B} y_n^2 + \frac{C}{\sqrt{AB}} y_{n-1}}$$

$$y_{n+1} = \frac{\sqrt{B}}{\sqrt{A}} \frac{\alpha + \beta\frac{\sqrt{A}}{\sqrt{B}} y_{n-1}}{A(1 + y_n^2 + C\frac{\sqrt{A}}{\sqrt{B}} y_{n-1})}$$

$$y_{n+1} = \frac{\alpha\frac{\sqrt{B}}{\sqrt{A^3}} + \frac{\beta}{A} y_{n-1}}{1 + y_n^2 + \frac{C}{\sqrt{AB}} y_{n-1}}.$$

Let $p = \alpha\frac{\sqrt{B}}{\sqrt{A^3}}$, $q = \frac{\beta}{A}$, and $r = \frac{C}{\sqrt{AB}}$. We get

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y_n^2 + ry_{n-1}}, \ n = 0,\ 1,\ 2,\ ...$$

3.2 Equilibrium points

In this section we prove the existence of the unique positive equilibrium point of the rational difference equation

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y_n^2 + ry_{n-1}}, \ n = 0,\ 1,\ 2,\ ... \tag{3.2.1}$$

with positive parameters $p,\ q,\ r$, and non-negative initial conditions. And we give a Matlab code to find it.

To find the equilibrium point, we solve the following equation

$$\bar{y} = \frac{p + q\bar{y}}{1 + \bar{y}^2 + r\bar{y}}$$

hence

$$\bar{y}^3 + r\bar{y}^2 + (1 - q)\bar{y} - p = 0. \tag{3.2.2}$$

To prove the existence of the positive fixed point we use the following theorem.

Theorem 3.1 (Descartes' rule of signs)**.** *Let $P(x)$ be a polynomial with real coefficients. Then the number of positive zeros of P is either equal to the number of variations in sign of $P(x)$ or less than this by an even number.*

By Descartes' rule of signs Equation (3.2.2) has one positive root, which is the unique positive equilibrium point of Equation (3.2.1).

To find the roots of Equation (3.2.2) we use the following code.

```
syms x q r p
t = [1 r (1-q) -p];
l= roots(t)
```

And then we choose the positive root to be $\bar{y}$.

3.3 Linearized equation

To find the linearized equation of (3.2.1) about the equilibrium point $\bar{y}$, let

$$f(x,y) = \frac{p+qy}{1+x^2+ry}$$
$$\frac{\partial f}{\partial x}(x,y) = \frac{-2x(p+qy)}{(1+x^2+ry)^2}$$
$$\frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-2\bar{y}(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}.$$

Since

$$\bar{y} = \frac{p+q\bar{y}}{1+\bar{y}^2+r\bar{y}}$$

we get

$$\frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}.$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = \frac{q(1+x^2+ry)-r(p+qy)}{(1+x^2+ry)^2}$$
$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q(1+\bar{y}^2+r\bar{y})-r(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}$$
$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}.$$

The linearized equation is

$$y_{n+1} = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}y_n + \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}y_{n-1}.$$

And the characteristic equation is

$$\lambda^2 + \frac{2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}\lambda - \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} = 0.$$

3.4 Local stability

To check when the unique positive equilibrium point $\bar{y}$ of Equation (3.2.1) is locally asymptotically stable, let

$$a = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}, \quad b = \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}$$

A sufficient condition for asymptotic stability of $\bar{y}$ is $|a| < 1-b < 2$. Which is equivalent to

$$-b < 1, \tag{3.4.1}$$
$$\textit{and} \;\; |a| < 1-b. \tag{3.4.2}$$

(3.4.1) always holds,

$$-\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} < 1$$

since

$$r\bar{y} - q < 1 + \bar{y}^2 + r\bar{y}$$

and hence,

$$1 + \bar{y}^2 + q > 0.$$

Which always holds.

And (3.4.2) is equivalent to

$$\frac{2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} < (1 + \frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}})$$

so

$$2\bar{y}^2 < 1 + \bar{y}^2 + r\bar{y} + r\bar{y} - q$$

and hence,

$$q < 1 - \bar{y}^2 + 2r\bar{y}.$$

So (3.4.2) holds when $q < 1 - \bar{y}^2 + 2r\bar{y}$.

Thus $q < 1 - \bar{y}^2 + 2r\bar{y}$ is a sufficient condition for asymptotic stability of $\bar{y}$.

3.5 Invariant intervals

Theorem 3.2. *Consider the difference equation (3.2.1), and $\{y_n\}_{n=-1}^{\infty}$ as a solution. Then the following are invariant intervals:*

1. $[0, q]$ *when* $r \geq 1,$ *and* $q \geq p$.

2. $[0, \frac{q}{r}]$ *when* $pr \leq q$.

Proof. 1. Assume that $r \geq 1,$ *and* $q \geq p$, and $y_{N-1},\ y_N \in [0, q]$ for some integer N.

$$\begin{aligned} y_{N+1} &= \frac{p + qy_{N-1}}{1 + y_N^2 + ry_{N-1}} \\ &\leq \frac{p + qy_{N-1}}{1 + ry_{N-1}} \\ &\leq \frac{q + qy_{N-1}}{1 + y_{N-1}}, \\ &= q \end{aligned}$$

And working inductively we complete the proof.

2. Assume that $pr \leq q$, and $y_{N-1},\ y_N \in [0, \frac{q}{r}]$ for some integer N.

$$\begin{aligned} y_{N+1} &= \frac{p + qy_{N-1}}{1 + y_N^2 + ry_{N-1}} \\ &= \frac{q(\frac{p}{q} + y_{N-1})}{r(\frac{1}{r} + \frac{1}{r}y_N^2 + y_{N-1})} \\ &\leq \frac{q(\frac{1}{r} + y_{N-1})}{r(\frac{1}{r} + y_{N-1})} \\ &= \frac{q}{r} \end{aligned}$$

And working inductively we complete the proof.

■

3.6 Boundedness

We will show that every solution of the difference equation (3.2.1) is bounded. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (3.2.1). then we have

$$\begin{aligned} &for\ n = 0,\ 1,\ 2,\ ... \\ 0 < y_{n+1} &= \frac{p+qy_{n-1}}{1+y_n^2+ry_{n-1}} \\ &= \frac{p}{1+y_n^2+ry_{n-1}} + \frac{qy_{n-1}}{1+y_n^2+ry_{n-1}} \\ &\leq \frac{p}{1} + \frac{qy_{n-1}}{ry_{n-1}} \\ &= p + \frac{q}{r}. \end{aligned}$$

Hence the solution is bounded, since it is bounded from below and from above.

3.7 Period two cycles

In general, we say that a solution $\{y_n\}_{n=-1}^{\infty}$ is of prime period two if the solution eventually takes the form:

$$...,\ \phi,\ \psi,\ \phi,\ \psi,\ ...$$

where ϕ *and* ψ are positive, and $\phi \neq \psi$.

Theorem 3.3. *Assume that Equation (3.2.1) has a two periodic cycle* $\{\phi,\ \psi\}$, *where* ϕ *and* ψ *are positive, and* $\phi \neq \psi$. *Then* q *must satisfy the following conditions:*

1.
$$q \leq 1 + r(\phi + \psi) \tag{3.7.1}$$

2.
$$q > 1 - \phi\psi \tag{3.7.2}$$

Proof: Assume $\{\phi,\ \psi\}$ is prime period two solution of Equation (3.2.1), then $\phi,\ \psi$ satisfy :

$$\phi = \frac{p+q\phi}{1+\psi^2+r\phi} \tag{3.7.3}$$

and

$$\psi = \frac{p + q\psi}{1 + \phi^2 + r\psi}. \tag{3.7.4}$$

From Equation (3.7.3) we have

$$\phi + \phi\psi^2 + r\phi^2 = p + q\phi, \tag{3.7.5}$$

and from Equation (3.7.4) we have

$$\psi + \psi\phi^2 + r\psi^2 = p + q\psi. \tag{3.7.6}$$

Subtracting Equation (3.7.6) from (3.7.5), we get:

$$(\phi - \psi) - \psi\phi(\phi - \psi) + r(\phi^2 - \psi^2) = q(\phi - \psi).$$

Since $\phi \neq \psi$, the last equation can be divided by $(\phi - \psi)$, and we get

$$1 - \psi\phi + r(\phi + \psi) = q. \tag{3.7.7}$$

So

$$\phi\psi = 1 + r(\phi + \psi) - q.$$

But $\psi\phi \geq 0$, so

$$1 + r(\phi + \psi) - q \geq 0,$$

hence

$$q \leq 1 + r(\phi + \psi).$$

Which is the first condition. From (3.7.7) we get also:

$$\phi + \psi = \frac{\phi\psi + q - 1}{r}.$$

But $\phi + \psi > 0$, so

$$\frac{\phi\psi + q - 1}{r} > 0,$$

since $r > 0$ we have

$$\phi\psi + q - 1 > 0,$$

hence

$$q > 1 - \phi\psi.$$

Which completes the proof.

■

To study the stability of the two cycle $\{\phi, \psi\}$ (if it exists), let

$$z_n = y_{n-1},$$

and

$$v_n = y_n.$$

We get the following system

$$z_{n+1} = v_n$$

$$v_{n+1} = \frac{p + qz_n}{1 + {v^2}_n + rz_n}, \ n = 0, \ 1, \ 2, \ ...$$

Let F be a function on $(0, \infty) \times (0, \infty)$ defined by

$$F\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{p+qz}{1+v^2+rz} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

is a fixed point of $F^2(z, v)$, where

$$F^2\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} F_1(v, z) \\ F_2(v, z) \end{pmatrix} = \begin{pmatrix} \frac{p+qz}{1+v^2+rz} \\ \frac{p+qv}{1+(F_1(v,z))^2+rv} \end{pmatrix}. \tag{3.7.8}$$

Now we find the Jacobian matrix of $F^2(z, v)$, we have

$$\frac{\partial F_1}{\partial z} = \frac{q(1 + v^2 + rz) - r(p + qz)}{(1 + v^2 + rz)^2},$$

$$\frac{\partial F_1}{\partial v} = \frac{-2(p + qz)v}{(1 + v^2 + rz)^2},$$

$$\frac{\partial F_2}{\partial z} = \frac{-2(p + qv)\frac{\partial F_1(z,v)}{\partial z}F_1(z, v)}{(1 + (F_1(v, z))^2 + rv)^2},$$

and

$$\frac{\partial F_2}{\partial v} = \frac{q(1+(F_1(v,z))^2+rv)-(p+qv)(2\frac{\partial F_1(z,v)}{\partial v}F_1(z,v)+r)}{(1+(F_1(v,z))^2+rv)^2}.$$

The Jacobian matrix is

$$JF^2(z,v) = \begin{pmatrix} \frac{q(1+v^2+rz)-r(p+qz)}{(1+v^2+rz)^2} & \frac{-2(p+qz)v}{(1+v^2+rz)^2} \\ \frac{-2(p+qv)\frac{\partial F_1(z,v)}{\partial z}F_1(z,v)}{(1+(F_1(v,z))^2+rv)^2} & \frac{q(1+(F_1(v,z))^2+rv)-(p+qv)(2\frac{\partial F_1(z,v)}{\partial v}F_1(z,v)+r)}{(1+(F_1(v,z))^2+rv)^2} \end{pmatrix}$$

$$JF^2(z,v)|_{(\phi,\psi)} = \begin{pmatrix} \frac{q-r(\phi}{(1+\psi^2+r\phi)} & \frac{-2\phi\psi}{(1+\psi^2+r\phi)} \\ \frac{-2(p+q\phi)\phi\psi}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)} & \frac{q(1+\psi^2+r\phi)+4\phi^2\psi^2-r\psi(1+\psi^2+r\phi)}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)} \end{pmatrix}$$

So

$$\det(JF^2(\phi,\psi)) = \frac{(q-r\phi)(q-r\psi)}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)},$$

and

$$tr(JF^2(\phi,\psi)) = \frac{(q-r\phi)(1+\phi^2+r\psi)+(q-r\psi)(1+\psi^2+r\phi)+4\phi^2\psi^2}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)}.$$

A sufficient condition for locally asymptotic stability of $\{\phi,\psi\}$ is

$$|tr(JF^2)| < 1 + det(JF^2) < 2.$$

Which is equivalent to

$$det(JF^2) < 1, \tag{3.7.9}$$

$$tr(JF^2) < 1 + det(JF^2), \tag{3.7.10}$$

$$-1 - det(JF^2) < tr(JF^2). \tag{3.7.11}$$

(3.7.9) holds when

$$(q-r\phi)(q-r\psi) < (1+\psi^2+r\phi)(1+\phi^2+r\psi),$$

So

$$(q-r\phi)(q-r\psi) - (1+\psi^2+r\phi)(1+\phi^2+r\psi) < 0.$$

And (3.7.10) holds when

$$\frac{(q-r\phi)(1+\phi^2+r\psi)+(q-r\psi)(1+\psi^2+r\phi)+4\phi^2\psi^2}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)} < 1+\frac{(q-r\phi)(q-r\psi)}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)},$$

so

$$(q-r\phi)(1+\phi^2+r\psi)+(q-r\psi)(1+\psi^2+r\phi)+4\phi^2\psi^2 < (1+\psi^2+r\phi)(1+\phi^2+r\psi)+(q-r\phi)(q-r\psi),$$

and hence

$$(q-r\phi)(1+\phi^2+r\psi)+(q-r\psi)(1+\psi^2+r\phi)+4\phi^2\psi^2 -(1+\psi^2+r\phi)(1+\phi^2+r\psi)-(q-r\phi)(q-r\psi) < 0.$$

(3.7.11) always holds since

$$\frac{(q-r\phi)(1+\phi^2+r\psi)+(q-r\psi)(1+\psi^2+r\phi)+4\phi^2\psi^2}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)} > -1-\frac{(q-r\phi)(q-r\psi)}{(1+\psi^2+r\phi)(1+\phi^2+r\psi)}$$

implies

$$-1-\phi^2-\psi^2-5\phi^2\psi^2-q^2-r^2\phi\psi-2q-q(\phi^2+\psi^2) < 0$$

which always holds.

3.8 Global stability

In this section we investigate a result about the global stability of the positive equilibrium point of (3.2.1) $\bar{y}$.

Theorem 3.4. *Assume* $pr \leq q \leq \frac{r\sqrt{r^2+4}-r^2}{2}$. *Then the positive equilibrium point* $\bar{y}$ *on the interval* $S=[0,\frac{q}{r}]$ *is globally asymptotically stable.*

Proof: this proof can easily be done depending on Theorem (1.7). Assume $pr \leq q$, and consider the function

$$f(x, y) = \frac{p + qy}{1 + x^2 + ry}.$$

Note that S is an invariant interval and all non-negative solutions of Equation (3.2.1) lie in this interval, and $f(x, y)$ on S is non-increasing function in x, and non-decreasing in y.

Now we need to show that the difference equation (3.2.1) has no solution of prime period two in S.

For seek of contradiction assume that the difference equation (3.2.1) has a solution of prime period two $\{\phi, \psi\} \in S$. Then q must satisfy

$$q > 1 - \phi\psi,$$

but since $\{\phi, \psi\} \in S$

$$1 - \phi\psi \geq 1 - \frac{q^2}{r^2},$$

hence

$$q > 1 - \frac{q^2}{r^2},$$

which is a contradiction, since $q \leq \frac{r\sqrt{r^2+4}-r^2}{2}$.

So Equation (3.2.1) has no solution of prime period two in S. Then both conditions of Theorem (1.7) hold, then (3.2.1) has a unique positive equilibrium point $\bar{y} \in S$, and it is globally asymptotically stable. ■

3.9 Matlab Codes and numerical discussion 1

In this section we introduce Matlab code that uses our results, and then we insert some examples.

Matlab code for finding the fixed point and its stability and solution behavior:

```
syms x;
r= ; %r value
p= ; %p value
q= ; %q value
t = [1 r (1-q) -p];
l= roots(t)
if l(1)>0
'The positive fixed point is'
y=l(1)
else if l(2)>0
'The positive fixed point is'
y=l(2)
else if l(3)>0
'The positive fixed point is'
y=l(3)
end
end
end
 U=(1+y^2+r*y)
a=(2*y^2)/(1+y^2+r*y)
b=(q-r*y)/(1+y^2+r*y)
if q<(1-y^2+2*r*y)
    if q<(1+3*y^2+2*r*y)
     'The positive fixed point is asymptoticlly stable'
    end
```

```
else
      'The positive fixed point is not asymptoticlly stable'
end
c=p*r
m=(((r*(r^2+4)^(0.5))-r^2)/2)
if q>=c && q<=m
     'The positive fixed point is globally asymptoticlly
        stable'
      n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(p+q*x(i-1))/(1+(x(i))^2+r*x(i-1));
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('The behavior of the solutions')
else
      'The positive fixed point is not globally
         asymptoticlly stable'
      n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(p+q*x(i-1))/(1+(x(i))^2+r*x(i-1));
```

```
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('The behavior of the solutions')
end
```

Example 3.1. *Consider the difference equation (3.2.1), take* $p = 4,\ q = 5,\ r = 0.5.$ *Equation (3.2.1) becomes*

$$y_{n+1} = \frac{4 + 5y_{n-1}}{1 + y_n^2 + 0.5y_{n-1}},\ n = 0,\ 1,\ 2,\ ...$$

With initial conditions $y_0 = 0.1,\ y_1 = 1.1.$

The theoretical positive equilibrium point will be $\bar{y} = 2.1786778129.$

Theoretically the positive equilibrium point $\bar{y}$ *is unstable since* $pr = 2 \leq q$ *but* $q > \frac{r\sqrt{r^2+4}-r^2}{2} = 0.3903882032.$
Figure (3.1) shows that the positive equilibrium point is unstable.

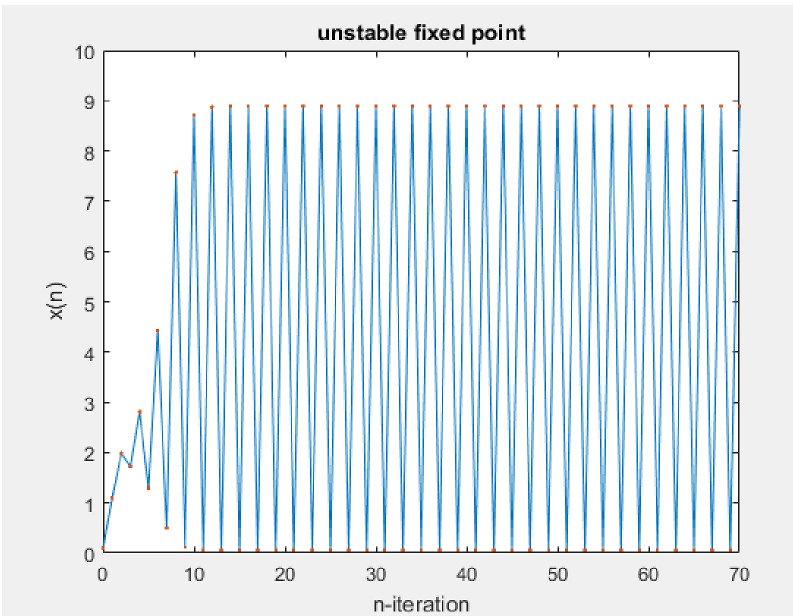


Fig. 3.1: The positive equilibrium point is unstable.

And using the Matlab code we get the following output

l =

2.1787 + 0.0000i

-1.3393 + 0.2053i

-1.3393 - 0.2053i

ans = 'The positive fixed point is'

y = 2.1787

U = 6.8360

a = 1.3887

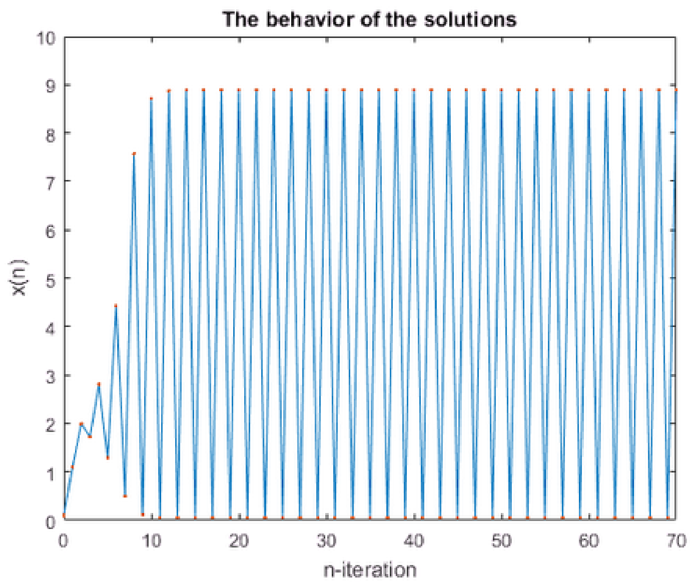
b = 0.5721

ans = 'The positive fixed point is not asymptoticly stable'

c = 2

m = 0.3904

ans = 'The positive fixed point is not globally asymptoticly stable'



Example 3.2. *Consider the difference equation (3.2.1), take* $p = 0.5,\ q = 0.3,\ r = 0.5$. *Equation (3.2.1) becomes*

$$y_{n+1} = \frac{0.5 + 0.3y_{n-1}}{1 + y_n^2 + 0.5y_{n-1}},\ n = 0,\ 1,\ 2,\ \ldots$$

With initial conditions $y_0 = 0.1,\ y_1 = 1.1$.

The theoretical positive equilibrium point will be $\bar{y} = 0.4457850401$.

Theoretically the positive equilibrium point $\bar{y}$ *is stable since* $pr = 0.25 \leq q$ *and* $q < \frac{r\sqrt{r^2+4}-r^2}{2} = 0.39$.
Figure (3.2) shows that the positive equilibrium point is stable.

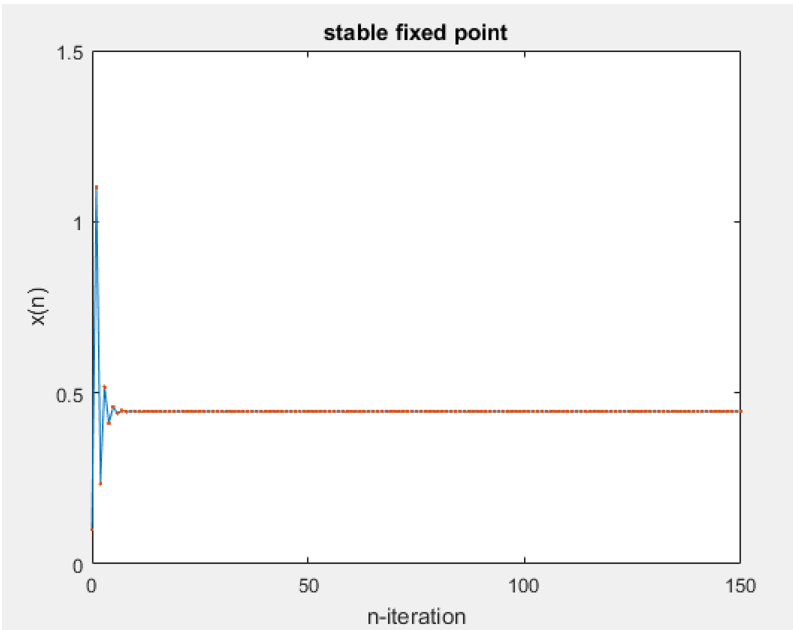


Fig. 3.2: The positive equilibrium point is stable.

And using the Matlab code we get the following output

l =

-0.4729 + 0.9476i

-0.4729 - 0.9476i

0.4458 + 0.0000i

ans = 'The positive fixed point is'

y = 0.4458

U = 1.4216

a = 0.2796

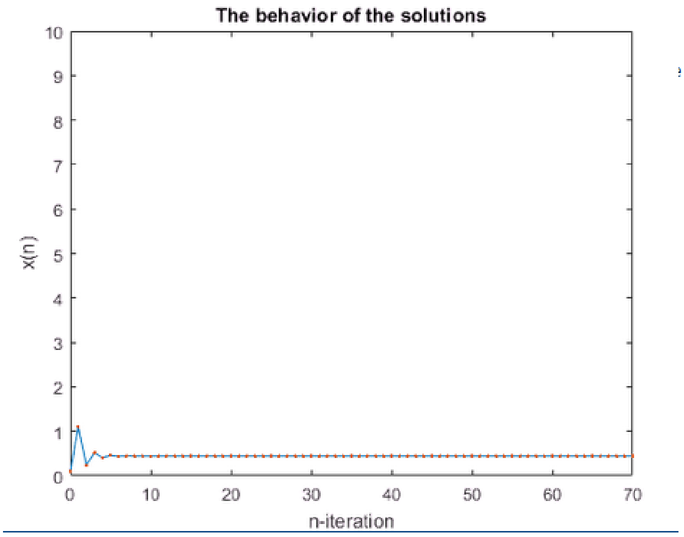
b = 0.0542

ans = 'The positive fixed point is asymptoticlly stable'

c = 0.2500

m = 0.3904

ans = 'The positive fixed point is globally asymptoticlly stable'



3.10 Bifurcation of $y_{n+1} = \frac{p+qy_{n-1}}{1+y_n^2+ry_{n-1}}$

In this section we study the types of bifurcation that occur at $q = q^*$ as q is the bifurcation parameter.

In order to convert Equation (3.2.1) to a second order dimensional system with three parameters p, q, and r, let

$$z_n = y_{n-1},$$

and

$$v_n = y_n.$$

We get the following system

$$z_{n+1} = v_n$$
$$v_{n+1} = \frac{p+qz_n}{1+v^2{}_n+rz_n},\ n = 0,\ 1,\ 2,\ ...$$

This system has the unique fixed point $(\bar{z},\bar{v})^T = (\bar{y},\bar{y})^T$. Convert this system in to second dimensional map

$$F\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} f_1(z,v) \\ f_2(z,v) \end{pmatrix} = \begin{pmatrix} v \\ \frac{p+qz}{1+v^2+rz} \end{pmatrix}. \tag{3.10.1}$$

Now we find the Jacobian matrix of $F(z,v)$, we have

$$\frac{\partial f_1}{\partial z} = 0,$$

$$\frac{\partial f_1}{\partial v} = 1,$$

$$\frac{\partial f_2}{\partial z} = \frac{q(1+v^2+rz)-r(p+qz)}{(1+v^2+rz)^2},$$

and

$$\frac{\partial f_2}{\partial v} = \frac{-2(p+qz)v}{(1+v^2+rz)^2}.$$

The Jacobian matrix is

$$JF(z,v) = \begin{pmatrix} 0 & 1 \\ \frac{q(1+v^2+rz)-r(p+qz)}{(1+v^2+rz)^2} & \frac{-2(p+qz)v}{(1+v^2+rz)^2} \end{pmatrix}$$

$$JF(z,v)|_{(\bar{y},\bar{y})} = \begin{pmatrix} 0 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} \end{pmatrix}$$

So

$$\det(JF(\bar{y},\bar{y})) = -\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}},$$

and

$$tr(JF(\bar{y},\bar{y})) = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}.$$

Theorem 3.5. *The fixed point $(\bar{y},\bar{y})$ of the system (3.10.1) undergoes a saddle-node bifurcation when $q = 2r\bar{y} + 3\bar{y}^2 + 1$.*

Proof: Saddle-node bifurcation happens when

$$\det(J) = tr(J) - 1.$$

So the fixed point $(\bar{y},\bar{y})$ of the system (3.10.1) undergoes a saddle-node bifurcation if

$$\det(JF(\bar{y},\bar{y})) = tr(JF(\bar{y},\bar{y})) - 1$$

or

$$-\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} = \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} - 1$$

so

$$-q + r\bar{y} = -2\bar{y}^2 - 1 - \bar{y}^2 - r\bar{y}$$

thus

$$q = 2r\bar{y} + 3\bar{y}^2 + 1.$$

So saddle-node bifurcation happens if $q = 2r\bar{y} + 3\bar{y}^2 + 1$. ■

Theorem 3.6. *The fixed point $(\bar{y},\bar{y})$ of the system (3.10.1) undergoes a period-doubling bifurcation when $q = 2r\bar{y} - \bar{y}^2 + 1$ if $r > \frac{\bar{y}^2-1}{2\bar{y}}$.*

Proof: Assume $r > \frac{\bar{y}^2-1}{2\bar{y}}$. Period-doubling bifurcation happens when

$$\det(J) = -tr(J) - 1.$$

So the fixed point $(\bar{y}, \bar{y})$ of the system (3.10.1) undergoes a period-doubling bifurcation if

$$\det(JF(\bar{y}, \bar{y})) = -tr(JF(\bar{y}, \bar{y})) - 1$$

or

$$-\frac{q - r\bar{y}}{1 + \bar{y}^2 + r\bar{y}} = -\frac{-2\bar{y}^2}{1 + \bar{y}^2 + r\bar{y}} - 1$$

so

$$-q + r\bar{y} = 2\bar{y}^2 - 1 - \bar{y}^2 - r\bar{y}$$

thus

$$q = 2r\bar{y} - \bar{y}^2 + 1.$$

Which is positive since $r > \frac{\bar{y}^2-1}{2\bar{y}}$. So period-doubling bifurcation happens if $q = 2r\bar{y} - \bar{y}^2 + 1$. ∎

Now consider Neimark-Sacker bifurcation which happens when

$$\det(J) = 1$$

and

$$-2 < tr(J) < 2.$$

So the system (3.10.1) undergoes Neimark-Sacker bifurcation when

$$\det(JF(\bar{y}, \bar{y})) = 1 \tag{3.10.2}$$

and

$$-2 < tr(JF(\bar{y}, \bar{y})) < 2.$$

Equation (3.10.2) holds if

$$-\frac{q - r\bar{y}}{1 + \bar{y}^2 + r\bar{y}} = 1$$

so

$$-q + r\bar{y} = 1 + \bar{y}^2 + r\bar{y}$$

thus

$$q = -(1 + \bar{y}^2).$$

Which is impossible since $q > 0$. So the system (3.10.1) can not undergo Neimark-Sacker bifurcation at $(\bar{y}, \bar{y})$.

3.11 Direction of The Period-Doubling (Flip) bifurcation

In this section we will find the direction of Flip bifurcation of system (3.10.1) at $q = 2r\bar{y} - \bar{y}^2 + 1$.

We need at first to shift the fixed point $(\bar{y}, \bar{y})$ to the origin. Let

$$w_n = z_n - \bar{y},\ u_n = v_n - \bar{y}.$$

System (3.10.1) will be

$$w_{n+1} = u_n$$

$$u_{n+1} = \frac{p + q(w_n + \bar{y})}{1 + (u_n + \bar{y})^2 + r(w_n + \bar{y})},\ n = 0,\ 1,\ 2,\ ...$$

Or

$$Y_{n+1} = AY_n + G(Y_n), \tag{3.11.1}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} \end{pmatrix},\ Y_n = \begin{pmatrix} w_n \\ u_n \end{pmatrix},$$

and

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\|Y\|^4)$$

$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \end{pmatrix} \ and\ C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \end{pmatrix}$$

where

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial\eta_k \partial\eta_j}|_{\eta=0}(x_k y_j)$$

and

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial\eta_l \partial\eta_k \partial\eta_j}|_{\eta=0}(x_l y_k z_j).$$

So $B_1(\psi,\phi) = 0$ and $C_1(\psi,\phi,\xi) = 0$,

$$B_2(\psi,\phi) = \frac{-2r(q - r\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}(\psi_1\phi_1) + \frac{2\bar{y}(2r\bar{y} - q)}{(1+\bar{y}^2+r\bar{y})^2}(\psi_1\phi_2 + \psi_2\phi_1) + \frac{8\bar{y}^3 - 2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}(\psi_2\phi_2),$$

and

$$C_2(\psi,\phi,\xi) = \frac{6r^2(q-r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}(\psi_1\phi_1\xi_1) + \frac{4r\bar{y}(2q-3r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}(\psi_1\phi_1\xi_2+\psi_1\phi_2\xi_1+\psi_2\phi_1\xi_1)$$
$$+\frac{2q(r\bar{y}+3\bar{y}^2-1)+4r(p-6\bar{y}^3)}{(1+\bar{y}^2+r\bar{y})^3}(\psi_2\phi_2\xi_1+\psi_2\phi_1\xi_2+\psi_1\phi_2\xi_2)$$
$$+\frac{20\bar{y}(p+q\bar{y})-48\bar{y}^4}{(1+\bar{y}^2+r\bar{y})^3}(\psi_2\phi_2\xi_2).$$

Now we find the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ at the bifurcation point $q = 2r\bar{y} - \bar{y}^2 + 1$.

Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ respectively. So we have

$$A\hat{q} = -\hat{q}, \text{ and } A^T p^* = -p^*.$$

Or

$$(A+I)\hat{q} = 0 \tag{3.11.2}$$
$$(A^T+I)p^* = 0. \tag{3.11.3}$$

Equation (3.11.2) is equivalent to

$$\begin{pmatrix} 1 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & 1+\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\hat{q}_1 = 1$, from the first equation we get

$$\hat{q}_1 + \hat{q}_2 = 0$$

so $\hat{q}_2 = -1$. Thus take $\hat{q} \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Note that in order to have a nonzero solution for $(A+I)\hat{q} = 0$, the matrix $(A+I)$ must be singular. Which means that $|A+I|$ must equal to zero, so

$$\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} + 1 - \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} = 0.$$

Thus, $\hat{q}$ satisfies the second equation $\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}\hat{q}_1 + (1 + \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}})\hat{q}_2 = 0$.

Now, consider Equation (3.11.3) which is equivalent to

$$\begin{pmatrix} 1 & \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} \\ 1 & 1 + \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} \end{pmatrix} \begin{pmatrix} {p^*}_1 \\ {p^*}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Take ${p^*}_2 = 1$, from the first equation we get

$${p^*}_1 + \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}}{p^*}_2 = 0$$

so ${p^*}_1 = \frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}}$. Thus take $p^* \sim \begin{pmatrix} \frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}} \\ 1 \end{pmatrix}$.

Note that in order to have a nonzero solution for $(A^T + I)p^* = 0$, the matrix $(A^T + I)$ must be singular. Which means that $|A^T + I|$ must equal to zero, so

$$\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} + 1 - \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} = 0.$$

Thus, p^* satisfies the second equation ${p^*}_1 + (1 + \frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}}){p^*}_2 = 0$.

Now, we normalize p^* and $\hat{q}$,

$$\langle p^*, \hat{q}\rangle = \frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}} - 1.$$

Take $\hat{p} = \eta \begin{pmatrix} \frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}} \\ 1 \end{pmatrix}$, $\eta = \frac{1}{\frac{r\bar{y}-q}{1+\bar{y}^2+r\bar{y}}-1} = -\frac{1+\bar{y}^2+r\bar{y}}{1+\bar{y}^2+q}$.

The critical eigenspace T^c corresponding to $\lambda = -1$ is one-dimensional and spanned by an eigenvector $\hat{q}$. Let T^{su} denote a one-dimensional linear eigenspace of A corresponding to all eigenvalues other than λ. Note that the matrix $(A - \lambda I_n)$ has common invariant spaces with the matrix A, so we conclude that $y \in T^{su}$ if and only if $\langle \hat{p}, y\rangle = 0$.

Any vector $x \in \mathbb{R}^n$ can be decomposed as

$$x = u\hat{q} + y$$

where $u\hat{q} \in T^c$, $y \in T^{su}$ and

$$u = \langle \hat{p}, x \rangle.$$

$$y = x - \langle \hat{p}, x \rangle \hat{q}.$$

In the coordinates (u, y), the map (3.11.1) can be written as

$$\tilde{u} = \lambda u + \langle \hat{p}, G(u\hat{q} + y) \rangle,$$

$$\tilde{y} = Ay + G(u\hat{q} + y) - \langle \hat{p}, G(u\hat{q} + y) \rangle \hat{q}.$$

Using Taylor expansions, the last two equations can be written as

$$\begin{aligned} \tilde{u} &= \lambda u + \frac{1}{2}\delta u^2 + u\langle b, y \rangle + \frac{1}{6}\sigma u^3 + ... \\ \tilde{y} &= Ay + \frac{1}{2}au^2 + ... \end{aligned} \tag{3.11.4}$$

where $u = \mathbb{R}$, $y \in \mathbb{R}^n$, δ, $\sigma \in \mathbb{R}$, a, $b \in \mathbb{R}^n$ and $\langle b, y \rangle = \sum_{i=1}^n b_i y_i$ is the standard scalar product, and can be expressed as

$$\langle b, y \rangle = \langle b, B(\hat{q}, y) \rangle.$$

The center manifold of (3.11.4) has the representation

$$y = V(u) = \frac{1}{2}w_2 u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset \mathbb{R}^n$, so that $\langle \hat{p}, w_2 \rangle = 0$. The vector w_2 satisfies

$$(A - I_n)w_2 + a = 0.$$

We have $\lambda = 1$ is not an eigenvalue of A, so the matrix $(A - I_n)$ is invertible in $\mathbb{R}^n$. Thus, we have

$$w_2 = -(A - I_n)^{-1}a$$

and the restriction of (3.11.4) to the center manifold takes the form

$$\tilde{u} = -u + \frac{1}{2}\delta u^2 + \frac{1}{6}(\sigma - 3\langle \hat{q}, (A - I_n)^{-1}a \rangle)u^3 + O(u^4)$$

where $\delta = \langle \hat{p}, B(\hat{q}, \hat{q}) \rangle$, $\sigma = \langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q}) \rangle$ and $a = B(\hat{q}, \hat{q}) - \langle \hat{p}, B(\hat{q}, \hat{q}) \rangle \hat{q}$.

Using the identity

$$(A - I_n)\hat{q} = \frac{1}{2}\hat{q},$$

the restricted map can br written as

$$\tilde{u} = -u + a(0)u^2 + b(0)u^3 + O(u^4) \tag{3.11.5}$$

where

$$a(0) = \frac{1}{2}\langle \hat{p}, B(\hat{q}, \hat{q})\rangle,$$

and

$$b(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q})\rangle - \frac{1}{4}(\langle \hat{p}, B(\hat{q}, \hat{q})\rangle)^2 - \frac{1}{2}\langle \hat{p}, B(\hat{q}, (A - I_n)^{-1}B(\hat{q}, \hat{q}))\rangle.$$

The map (3.11.5) can be transformed to the normal form

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4)$$

where

$$c(0) = a^2(0) - b(0).$$

Thus,the critical normal form coefficient $c(0)$, allows us to predict the direction of bifurcation of the period-two cycle. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q})\rangle - \frac{1}{2}\langle \hat{p}, B(\hat{q}, (A - I_n)^{-1}B(\hat{q}, \hat{q}))\rangle.$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q = 2r\bar{y} - \bar{y}^2 + 1$.

$$B(\hat{q}, \hat{q}) = \begin{pmatrix} 0 \\ \frac{-2r(q-r\bar{y})-4\bar{y}(2r\bar{y}-q)+8\bar{y}^3-2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2} \end{pmatrix}.$$

$$C(\hat{q}, \hat{q}, \hat{q}) = \begin{pmatrix} 0 \\ \frac{6r^2(q-r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3} - \frac{12r\bar{y}(2q-3r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3} + 3\frac{2q(r\bar{y}+3\bar{y}^2-1)+4r(p-6\bar{y}^3)}{(1+\bar{y}^2+r\bar{y})^3} - \frac{20\bar{y}(p+q\bar{y})-48\bar{y}^4}{(1+\bar{y}^2+r\bar{y})^3} \end{pmatrix}.$$

$$\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle = -\left(\frac{1+\bar{y}^2+r\bar{y}}{1+\bar{y}^2+q}\right)\left[\frac{6r^2(q-r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3} - \frac{12r\bar{y}(2q-3r\bar{y})}{(1+\bar{y}^2+r\bar{y})^3}\right.$$
$$\left. +3\frac{2q(r\bar{y}+3\bar{y}^2-1)+4r(p-6\bar{y}^3)}{(1+\bar{y}^2+r\bar{y})^3} - \frac{20\bar{y}(p+q\bar{y})-48\bar{y}^4}{(1+\bar{y}^2+r\bar{y})^3}\right].$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & -1+\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} \end{pmatrix}^{-1} = \frac{1+\bar{y}^2+r\bar{y}}{2\bar{y}^2}\begin{pmatrix} -1+\frac{-2\bar{y}^2}{1+\bar{y}^2+r\bar{y}} & -1 \\ -\frac{q-r\bar{y}}{1+\bar{y}^2+r\bar{y}} & -1 \end{pmatrix}.$$

$$(A-I)^{-1}B(\hat{q},\hat{q}) = \frac{1+\bar{y}^2+r\bar{y}}{2\bar{y}^2}\begin{pmatrix} \frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2} \\ \frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2} \end{pmatrix}.$$

$$B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q})) = \frac{1+\bar{y}^2+r\bar{y}}{2\bar{y}^2}\begin{pmatrix} 0 \\ m \end{pmatrix},$$

where

$$m = \left(\frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\right)\left(\frac{-2r(q-r\bar{y})-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\right).$$

$$\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle = \left(\left[\frac{2r(q-r\bar{y})+4\bar{y}(2r\bar{y}-q)-8\bar{y}^3+2(p+q\bar{y})}{2\bar{y}^2(1+\bar{y}^2+q)}\right]\right.$$
$$\left.\left[\frac{-2r(q-r\bar{y})-8\bar{y}^3+2(p+q\bar{y})}{(1+\bar{y}^2+r\bar{y})^2}\right]\right).$$

3.12 Matlab Codes and numerical discussion 2

In this section we introduce Matlab code that uses our results, and then we insert an example.

Matlab code for period-doubling bifurcation:

```
r=  ; %r value
p=  ; %p value
a=-1*r;
u=0;
t = [2 a 0 -p];
l= roots(t)
for k=1:3
if l(k)>0
     y=l(k);
     if (2*r*y-y^2+1)>0
     u=u+1;
     'The positive fixed point is'
y=l(k)
'The bifurcation valu of the parameter q is '
q=(2*r*y-y^2+1)
A=-1*( (1+y^2+r*(y))/(1+y^2+q));
B=6*(r^2)*(q-r*y)-12*r*y*((2*q-3*r*y))+3*(2*q*(r*y+3*(y^2)
    -1)+4*r*(p-6*(y^3)))-(20*y*(p+q*y)-48*y^4);
 U=(1+y^2+r*y)^3 ;
D=A*(B/U);
F=(2*r*(q-r*y)+4*y*(2*r*y-q)-8*y^3+2*(p+q*y))/((2*y^2 )*(1+y
    ^2+q )  );
L=(-2*r*(q-r*y)-8*y^3+2*(p+q*y))/((1+y^2+r*y)^2 );
J=F*L;
c=(1/6)*D-(0.5 )*J
if c>0
```

```
    'A unique and stable period-two cycle bifurcates from
       the fixed point at the bifurcation point.'
end
amin=0;
amax=10;
x0=.2;x1=.3;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
x(i+1)=(p+a.*x(i-1))/(1+(x(i))^2+r*x(i-1));
if (i>750)
z(j,i-750)=x(i+1);
end
end
end
plot(t,z,'blue','MarkerSize',5),title ('Period-doubling
    bifurcation')
     end
end
end

if u==0
```

```
'The system does not undergo period-doubling (flip )
    bifurcation. '
```
end

Example 3.3. *Consider the difference equation (3.2.1). Fix p, r, and consider q as bifurcation parameter. Take* $p = 1$, $r = 0.9$, *and* $0 < q \leq 10$. *Equation (3.2.1) becomes*

$$y_{n+1} = \frac{1 + qy_{n-1}}{1 + y_n^2 + 0.9y_{n-1}}, \quad n = 0,\ 1,\ 2,\ \ldots \tag{3.12.1}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{1+qy_1(n)}{1+y_2(n)^2{}_n+0.9y_1(n)} \end{pmatrix}. \tag{3.12.2}$$

The positive equilibrium point $\bar{y}$ *of (3.12.1) satisfies*

$$\bar{y}^3 + 0.9\bar{y}^2 + (1-q)\bar{y} - 1 = 0. \tag{3.12.3}$$

Theorem 3.6 shows that the fixed point undergoes a period-doubling bifurcation at $q^* = 1.8\bar{y} - \bar{y}^2 + 1$. *So Equation (3.12.3) at* q^* *becomes*

$$2\bar{y}^3 - 0.9\bar{y}^2 - 1 = 0.$$

Thus the theoretical fixed point of (3.12.1) is

$$\bar{y} = 0.97546665.$$

Note that $r = 0.9 > \frac{\bar{y}^2-1}{2\bar{y}} = -0.0236$, *so the condition of Theorem 3.6 holds. Substituting the value of* $\bar{y}$ *in* q^* *we get*

$$q^* = 1.8043047.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$\hat{q} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \hat{p} = (-0.7533482)\begin{pmatrix} -0.2466518 \\ 1 \end{pmatrix}.$$

$$c(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle - \frac{1}{2}\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle.$$

$$\langle \hat{p}, C(\hat{q},\hat{q},\hat{q})\rangle = 1.94576.$$

$$\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle = 0.0266652827.$$

So

$$c(0) = 0.1857633587 > 0$$

So this shows that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^* = 1.8043047$. *Figure (3.3) shows the stable period-two cycle.*

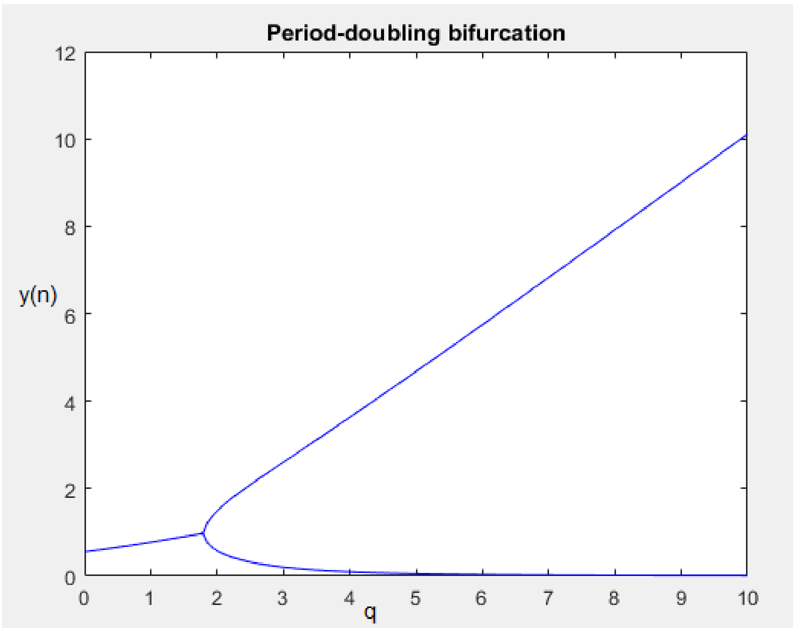


Fig. 3.3: Period-doubling bifurcation of $y_{n+1} = \frac{1+qy_{n-1}}{1+y_n^2+0.9y_{n-1}}$.

And using the Matlab code we get the following output

l =

0.9755 + 0.0000i

-0.2627 + 0.6660i

-0.2627 - 0.6660i

ans = 'The positive fixed point is'

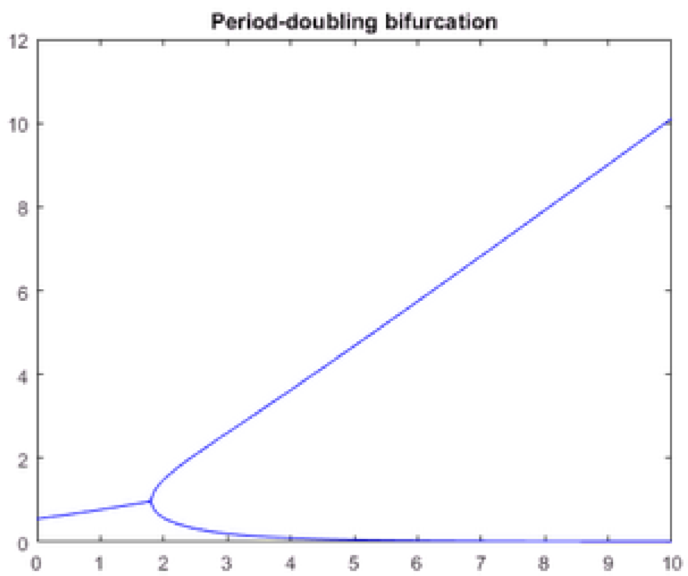
y = 0.9755

ans = 'The bifurcation valu of the parameter q is '

q = 1.8043

c = 0.1858

ans = ' A unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point.'



4. DYNAMICS OF $X_{N+1} = \frac{\alpha+\beta X_{N-1}^2}{A+BX_N+CX_{N-1}^2}$

In this chapter we consider the second order, quadratic rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}^2}{A + Bx_n + Cx_{n-1}^2}, \;\; n = 0,\; 1,\; 2,\; \dots \tag{4.0.1}$$

with positive parameters $\alpha,\; \beta,\; A,\; B,\; C$, and non-negative initial conditions.

We will focus on the dynamic behavior of the positive fixed point and the type of bifurcation exist where the change of stability occurs. And then we introduce some Matlab codes that use our results.

4.1 Change of variables

Consider Equation (4.0.1), let

$$x_n = \frac{A}{B} y_n.$$

Then

$$x_{n+1} = \frac{A}{B} y_{n+1},$$

and

$$x_{n-1} = \frac{A}{B} y_{n-1}.$$

So Equation (4.0.1) becomes

$$\frac{A}{B} y_{n+1} = \frac{\alpha + \beta \frac{A^2}{B^2} y_{n-1}^2}{A + B\frac{A}{B} y_n + C\frac{A^2}{B^2} y_{n-1}^2}$$

$$y_{n+1} = \frac{B}{A}\frac{\alpha + \beta\frac{A^2}{B^2}y_{n-1}^2}{A(1 + y_n + C\frac{A}{B^2}y_{n-1}^2)}$$

$$y_{n+1} = \frac{\alpha\frac{B}{A^2} + \frac{\beta}{B}y_{n-1}^2}{1 + y_n + \frac{CA}{B^2}y_{n-1}^2}.$$

Let $p = \alpha\frac{B}{A^2}$, $q = \frac{\beta}{B}$, and $r = \frac{CA}{B^2}$. We get

$$y_{n+1} = \frac{p + qy_{n-1}^2}{1 + y_n + ry_{n-1}^2}, \ n = 0, \ 1, \ 2, \ ...$$

4.2 Equilibrium points

In this section we prove the existence of a unique positive equilibrium point of the rational difference equation

$$y_{n+1} = \frac{p + qy_{n-1}^2}{1 + y_n + ry_{n-1}^2}, \ n = 0, \ 1, \ 2, \ ... \tag{4.2.1}$$

with positive parameters $p, \ q, \ r$, and non-negative initial conditions. And we give a Matlab code to find it. To find the equilibrium point, we solve the following equation

$$\bar{y} = \frac{p + q\bar{y}^2}{1 + \bar{y} + r\bar{y}^2}$$

hence

$$r\bar{y}^3 + (1 - q)\bar{y}^2 + \bar{y} - p = 0. \tag{4.2.2}$$

can be considered as two curves with behavior

$$\underbrace{r\bar{y}^2 + (1 - q)\bar{y}}_{\text{a parabola}} = \underbrace{\frac{p}{\bar{y}} - 1}_{\text{a hyperbola}}.$$

Equation (4.2.1) has a unique positive equilibrium point $\bar{y}$, which can be obtained as an intersection point of these two curves. From figures 4.1 and 4.2 we obtain the required conclusion.

To find the roots of Equation (4.2.2) we use the following code.

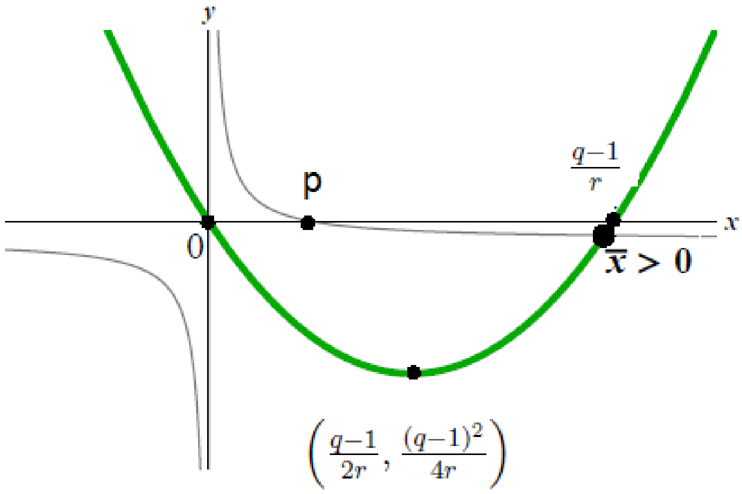


Fig. 4.1: The equilibrium of (4.2.1), $q > 1$.

```
syms x q r p
t = [r (1-q) 1 -p];
l= roots(t)
```

And then we choose the positive root to be $\bar{y}$.

4.3 Linearized equation

To find the linearized equation of (4.2.1) about the equilibrium point $\bar{y}$, let

$$f(x,y) = \frac{p+qy^2}{1+x+ry^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{-(p+qy^2)}{(1+x+ry^2)^2}$$

$$\frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-(p+q\bar{y}^2)}{(1+\bar{y}+r\bar{y}^2)^2}.$$

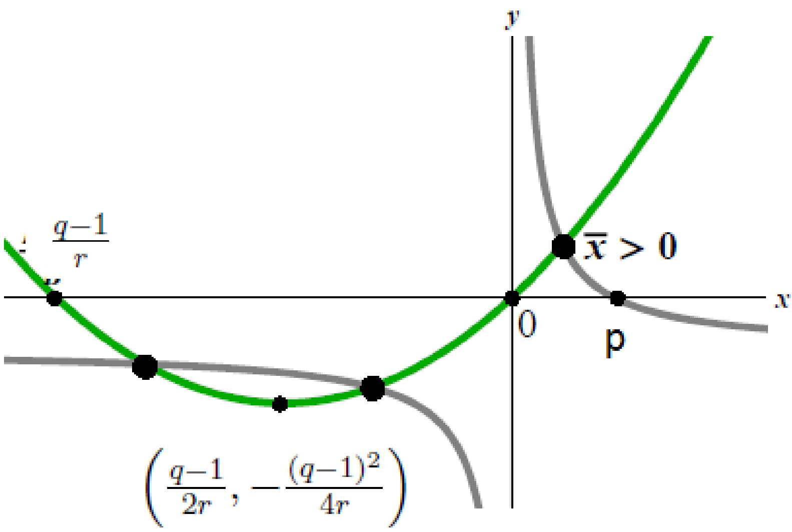


Fig. 4.2: The equilibrium of (4.2.1), $0 < q < 1$.

Since

$$\bar{y} = \frac{p + q\bar{y}^2}{1 + \bar{y} + r\bar{y}^2}$$

we get

$$\frac{\partial f}{\partial x}(\bar{y}, \bar{y}) = \frac{-\bar{y}}{1 + \bar{y} + r\bar{y}^2}.$$

Similarly,

$$\frac{\partial f}{\partial y}(x, y) = \frac{2qy(1 + x + ry^2) - 2ry(p + qy^2)}{(1 + x + ry^2)^2}$$

$$\frac{\partial f}{\partial y}(\bar{y}, \bar{y}) = \frac{2q\bar{y}(1 + \bar{y} + r\bar{y}^2) - 2r\bar{y}(p + q\bar{y})}{(1 + \bar{y} + r\bar{y}^2)^2}$$

$$\frac{\partial f}{\partial y}(\bar{y}, \bar{y}) = \frac{2\bar{y}(q - r\bar{y})}{1 + \bar{y} + r\bar{y}^2}.$$

The linearized equation is

$$y_{n+1} = \frac{-\bar{y}}{1 + \bar{y} + r\bar{y}^2} y_n + \frac{2\bar{y}(q - r\bar{y})}{1 + \bar{y} + r\bar{y}^2} y_{n-1}.$$

And the characteristic equation is

$$\lambda^2 + \frac{\bar{y}}{1 + \bar{y} + r\bar{y}^2}\lambda - \frac{2\bar{y}(q - r\bar{y})}{1 + \bar{y} + r\bar{y}^2} = 0.$$

4.4 Local stability

To check when the unique positive equilibrium point $\bar{y}$ of Equation (4.2.1) is locally asymptotically stable, let

$$a = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}, \quad b = \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}$$

A sufficient condition for asymptotic stability of $\bar{y}$ is $|a| < 1-b < 2$. Which is equivalent to

$$-b < 1, \tag{4.4.1}$$

$$\text{and } |a| < 1-b. \tag{4.4.2}$$

(4.4.1) holds when

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} < 1$$

so

$$2r\bar{y}^2 - 2\bar{y}q < 1+\bar{y}+r\bar{y}^2$$

and hence,

$$q > \frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}}.$$

Equation (4.4.2) is equivalent to

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} < 1 + \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}$$

so

$$\bar{y} < -2\bar{y}q + 3r\bar{y}^2 + 1 + \bar{y}$$

and hence,

$$q < \frac{1+3r\bar{y}^2}{2\bar{y}}.$$

So (4.4.2) holds when $q < \frac{1+3r\bar{y}^2}{2\bar{y}}$. Hence a sufficient condition for asymptotic stability of $\bar{y}$ is

$$\frac{-1+r\bar{y}^2-\bar{y}}{2\bar{y}} < q < \frac{1+3r\bar{y}^2}{2\bar{y}}.$$

4.5 Invariant intervals

Theorem 4.1. *Consider the difference equation (4.2.1), and let* $\{y_n\}_{n=-1}^{\infty}$ *be a solution. Then* $[0, \frac{q}{r}]$ *when* $pr \leq q$ *is an invariant interval.*

Proof. Assume that $pr \leq q$, and $y_{N-1},\ y_N \in [0, \frac{q}{r}]$ for some integer N.

$$\begin{aligned} y_{N+1} &= \frac{p+qy_{N-1}^2}{1+y_N+ry_{N-1}^2} \\ &= \frac{q(\frac{p}{q}+y_{N-1}^2)}{r(\frac{1}{r}+\frac{1}{r}y_N+y_{N-1}^2)} \\ &\leq \frac{q(\frac{1}{r}+y_{N-1}^2)}{r(\frac{1}{r}+y_{N-1}^2)} \\ &= \frac{q}{r} \end{aligned}$$

And working inductively we complete the proof.

∎

4.6 Boundedness

We will show that every solution of the difference equation (4.2.1) is bounded. Let $\{y_n\}_{n=-1}^{\infty}$ be a solution of (4.2.1), then we have for $n = 0,\ 1,\ 2,\ ...$

$$\begin{aligned} 0 < y_{n+1} &= \frac{p + qy_{n-1}^2}{1 + y_n + ry_{n-1}^2} \\ &= \frac{p}{1 + y_n + ry_{n-1}^2} + \frac{qy_{n-1}^2}{1 + y_n + ry_{n-1}^2} \\ &\leq \frac{p}{1} + \frac{qy_{n-1}^2}{ry_{n-1}^2} \\ &= p + \frac{q}{r}. \end{aligned}$$

Hence the solution is bounded, since it is bounded from below and from above.

4.7 Period two cycles

In general, we say that the the difference equation has a prime period two solution if the solution eventually takes the form:

$$...,\ \phi,\ \psi,\ \phi,\ \psi,\ ...$$

where ϕ *and* ψ are positive, and $\phi \neq \psi$.

Theorem 4.2. *Assume that Equation (4.2.1) has a two periodic cycle $\{\phi,\ \psi\}$, where ϕ and ψ are positive, and $\phi \neq \psi$. Then q must satisfy the following condition:*

$$q > \frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi}. \tag{4.7.1}$$

Proof. Assume $\{\phi,\ \psi\}$ is prime period two solution of Equation (4.2.1), then $\phi,\ \psi$ satisfy :

$$\phi = \frac{p + q\phi^2}{1 + \psi + r\phi^2} \tag{4.7.2}$$

and

$$\psi = \frac{p + q\psi^2}{1 + \phi + r\psi^2}. \tag{4.7.3}$$

From Equation (4.7.2) we have

$$\phi + \phi\psi + r\phi^3 = p + q\phi^2, \tag{4.7.4}$$

and from Equation (4.7.3) we have

$$\psi + \psi\phi + r\psi^3 = p + q\psi^2. \tag{4.7.5}$$

Subtracting Equation (4.7.5) from (4.7.4), we get:

$$(\phi - \psi) + r(\phi^3 - \psi^3) = q(\phi^2 - \psi^2).$$

Since $\phi \neq \psi$, the last equation can be divided by $(\phi - \psi)$, and we get

$$1 + r(\phi^2 + \phi\psi + \psi^2) = q(\phi + \psi). \tag{4.7.6}$$

So

$$\phi\psi = \frac{-1 - r(\phi^2 + \psi^2) + q(\phi + \psi)}{r}.$$

But $\psi\phi \geq 0$, so

$$-1 - r(\phi^2 + \psi^2) + q(\phi + \psi) \geq 0,$$

hence

$$q > \frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi}.$$

Which completes the proof. Note that from (4.7.6) we get also:

$$\phi + \psi = \frac{r(\phi^2 + \phi\psi + \psi^2) + 1}{q}.$$

Which is always positive. ∎

To study the stability of the two cycle $\{\phi, \psi\}$ (if it exists), let

$$z_n = y_{n-1},$$

and

$$v_n = y_n.$$

We get the following system

$$z_{n+1} = v_n$$

$$v_{n+1} = \frac{p + q{z^2}_n}{1 + v_n + r{z^2}_n}, \ n = 0, \ 1, \ 2, \ ...$$

Let F be a function on $(0, \infty) \times (0, \infty)$ defined by

$$F\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{p+qz^2}{1+v+rz^2} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

is a fixed point of $F^2(z, v)$, where

$$F^2\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} F_1(v,z) \\ F_2(v,z) \end{pmatrix} = \begin{pmatrix} \frac{p+qz^2}{1+v+rz^2} \\ \frac{p+qv^2}{1+(F_1(v,z))+rv^2} \end{pmatrix}. \tag{4.7.7}$$

Now we find the Jacobian matrix of $F^2(z, v)$, we have

$$\frac{\partial F_1}{\partial z} = \frac{(1+v+rz^2)2qz - (p+qz^2)(2rz)}{(1+v+rz^2)^2},$$

$$\frac{\partial F_1}{\partial v} = \frac{-(p+qz^2)}{(1+v+rz^2)^2},$$

$$\frac{\partial F_2}{\partial z} = \frac{-(p+qv^2)\frac{\partial F_1(z,v)}{\partial z}}{(1+(F_1(v,z))+rv^2)^2},$$

and

$$\frac{\partial F_2}{\partial v} = \frac{(1+(F_1(v,z))+rv^2)(2qv) - (p+qv^2)(\frac{\partial F_1(z,v)}{\partial v} + 2rv)}{(1+(F_1(v,z))+rv^2)^2}.$$

The Jacobian matrix is

$$JF^2(z,v) = \begin{pmatrix} \frac{(1+v+rz^2)2qz-(p+qz^2)(2rz)}{(1+v+rz^2)^2} & \frac{-(p+qz^2)}{(1+v+rz^2)^2} \\ \frac{-(p+qv^2)\frac{\partial F_1(z,v)}{\partial z}}{(1+(F_1(v,z))+rv^2)^2} & \frac{(1+(F_1(v,z))+rv^2)(2qv)-(p+qv^2)(\frac{\partial F_1(z,v)}{\partial v}+2rv)}{(1+(F_1(v,z))+rv^2)^2} \end{pmatrix}$$

$$JF^2(z,v)|_{(\phi,\psi)} = \begin{pmatrix} \frac{2\phi(q-r\phi)}{(1+\psi+r\phi^2)} & \frac{-\phi}{(1+\psi+r\phi^2)} \\ \frac{-2\phi\psi(p-2r\phi)}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)} & \frac{(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)} \end{pmatrix}$$

So

$$\det(JF^2(\phi,\psi)) = \frac{2\phi\psi(q-2r\phi)(q-2r\psi)}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)},$$

and

$$tr(JF^2(\phi,\psi)) = \frac{2\phi(q-2r\phi)(1+\phi+r\psi^2)+(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)}.$$

A sufficient condition for locally asymptotic stability of $\{\phi,\psi\}$ is

$$|tr(JF^2)| < 1 + det(JF^2) < 2.$$

Which is equivalent to

$$det(JF^2) < 1, \tag{4.7.8}$$

$$tr(JF^2) < 1 + det(JF^2), \tag{4.7.9}$$

$$-1 - det(JF^2) < tr(JF^2). \tag{4.7.10}$$

(4.7.8) holds when

$$2\phi\psi(q-2r\phi)(q-2r\psi) < (1+\psi+r\phi^2)(1+\phi+r\psi^2),$$

So

$$2\phi\psi(q-2r\phi)(q-2r\psi) - (1+\psi+r\phi^2)(1+\phi+r\psi^2) < 0.$$

And (4.7.9) holds when

$$\frac{2\phi(q-2r\phi)(1+\phi+r\psi^2)+(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)} < 1 + \frac{2\phi\psi(q-2r\phi)(q-2r\psi)}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)},$$

so

$$2\phi(q-2r\phi)(1+\phi+r\psi^2)+(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi < (1+\psi+r\phi^2)(1+\phi+r\psi^2)+2\phi\psi(q-2r\phi)(q-2r\psi),$$

and hence

$$2\phi(q-2r\phi)(1+\phi+r\psi^2)+(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi$$
$$-(1+\psi+r\phi^2)(1+\phi+r\psi^2)-2\phi\psi(q-2r\phi)(q-2r\psi)<0.$$

(4.7.10) holds when

$$\frac{2\phi(q-2r\phi)(1+\phi+r\psi^2)+(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)}$$
$$>-1-\frac{2\phi\psi(q-2r\phi)(q-2r\psi)}{(1+\psi+r\phi^2)(1+\phi+r\psi^2)},$$

so

$$2\phi(q-2r\phi)(1+\phi+r\psi^2)+(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi$$
$$>-(1+\psi+r\phi^2)(1+\phi+r\psi^2)-2\phi\psi(q-2r\phi)(q-2r\psi),$$

and hence

$$2\phi(q-2r\phi)(1+\phi+r\psi^2)+(2q\psi-2r\psi^2)(1+\psi+r\phi^2)+\phi\psi$$
$$+(1+\psi+r\phi^2)(1+\phi+r\psi^2)+2\phi\psi(q-2r\phi)(q-2r\psi)>0.$$

4.8 Global stability

In this section we investigate a result about the global stability of the positive equilibrium point of (4.2.1) $\bar{y}$.

Theorem 4.3. *Assume $pr \le q \le \frac{\sqrt{r}}{\sqrt{2}}$. Then the positive equilibrium point $\bar{y}$ on the interval $S = [0, \frac{q}{r}]$ is globally asymptotically stable.*

Proof: this proof can easily done depending on Theorem (1.7). Assume $pr \le q$, and consider the function

$$f(x,y) = \frac{p+qy^2}{1+x+ry^2}.$$

Note that S is an invariant interval and all non-negative solutions of Equation (4.2.1) lie in this interval, and $f(x, y)$ on S is non-increasing function in x, and non-decreasing in y.

Now we need to show that the difference equation (4.2.1) has no solution of prime period two in S.

For seek of contradiction assume that the difference equation (4.2.1) has a solution of prime period two $\{\phi, \psi\} \in S$. Then q must satisfy

$$q > \frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi},$$

but since $\{\phi, \psi\} \in S$

$$\frac{1 + r(\phi^2 + \psi^2)}{\phi + \psi} \geq \frac{1 + 0}{\frac{q}{r} + \frac{q}{r}},$$

hence

$$q > \frac{r}{2q},$$

so

$$q^2 > \frac{r}{2},$$

which is a contradiction, since $q \leq \frac{\sqrt{r}}{\sqrt{2}}$.

So Equation (4.2.1) has no solution of prime period two in S. Then both conditions of Theorem (1.7) hold, then (4.2.1) has a unique positive equilibrium point $\bar{y} \in S$, and it is globally asymptotically stable. ■

4.9 Matlab codes and numerical discussion 1

In this section we introduce Matlab codes that use our results, and then we insert some examples.

Matlab code for finding the fixed point and its stability and solution behavior:

```
 syms x;
r=  ; % r value
p=  ; % p value
q=  ; % q value
t = [r (1-q) 1 -p];
l= roots(t)
if l(1)>0
'The positive fixed point is'
y=l(1)
else if l(2)>0
'The positive fixed point is'
y=l(2)
else if l(3)>0
'The positive fixed point is'
y=l(3)
end
end
end
a=(y)/(1+y+r*y^2)
b= (2*y*(q-r*y))/(1+y+r*y^2)
if q<((1+3*r*y^2)/(2*y)) && q>((-1+r*y^2-y)/(2*y)) && q
   <((1+3*r*y^2 +2*y)/(2*y))
     'The positive fixed point is asymptoticlly stable'
else
      'The positive fixed point is not asymptoticlly stable'
end
c=p*r
m=(((r)^(0.5))/(2)^(0.5))
if q>=c && q<=m
     'The positive fixed point is globally asymptoticlly
        stable'
```

```
        n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(p+q*(x(i-1))^2)/(1+(x(i))+r*(x(i-1))^2);
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('The behavior of the solutions')
else
         'The positive fixed point is not globally
            asymptoticlly stable'
        n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(p+q*(x(i-1))^2)/(1+(x(i))^2+r*(x(i-1))^2);
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('The behavior of the solutions')
end
```

Example 4.1. *Consider the difference equation (4.2.1), take* $p = 0.4,\ q = 5,\ r =$

0.5. *Equation (4.2.1) becomes*

$$y_{n+1} = \frac{0.4 + 5y_{n-1}^2}{1 + y_n + 0.5y_{n-1}^2}, \quad n = 0, \ 1, \ 2, \ \dots$$

With initial conditions $y_0 = 0.1$, $y_1 = 1.1$.

The theoretical positive equilibrium point will be $\bar{y} = 7.7554165829$.

Theoretically the positive equilibrium point $\bar{y}$ *is unstable since* $pr = 0.12 \leq q$ *but* $q > \frac{\sqrt{r}}{\sqrt{2}} = 0.5$.

Figure (4.3) shows that the positive equilibrium point is unstable.

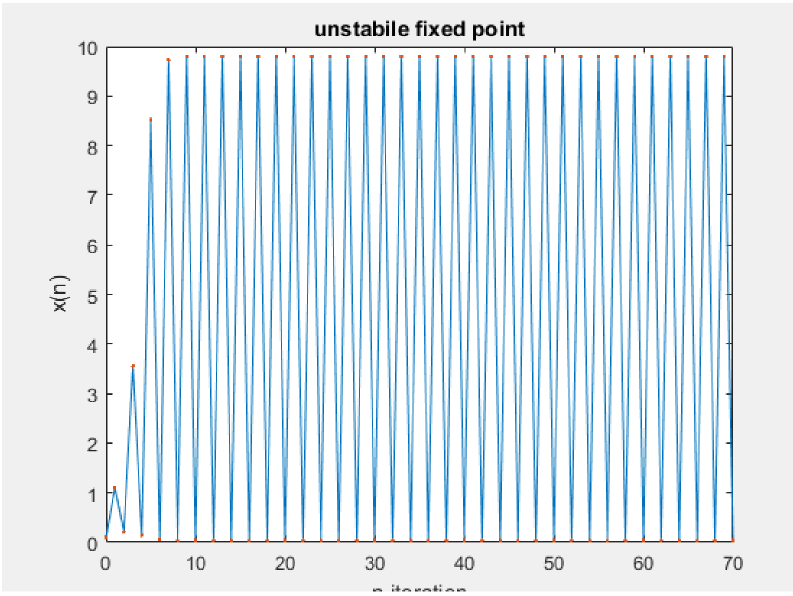


Fig. 4.3: The positive equilibrium point is unstable.

And using the Matlab code we get the following output

l =

7.8750 + 0.0000i

0.0625 + 1.0060i

0.0625 - 1.0060i

ans = 'The positive fixed point is'

y = 7.8750

a = 0.1975

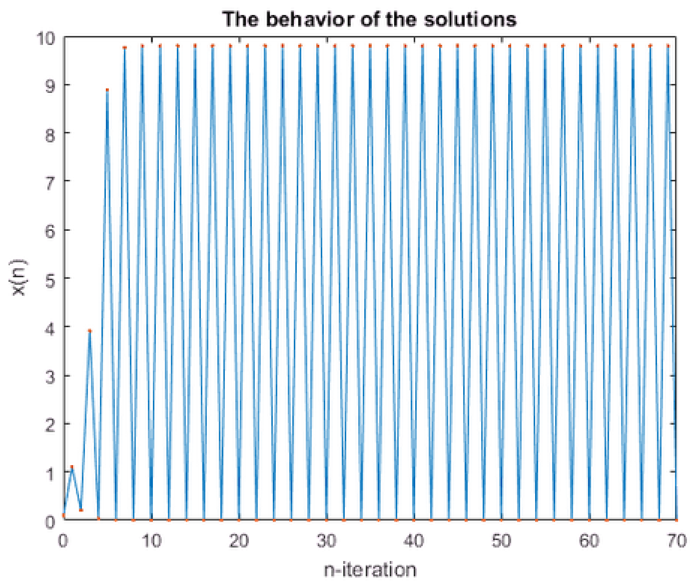
b = 0.4196

ans = 'The positive fixed point is asymptoticly stable'

c = 2

m = 0.5000

ans = 'The positive fixed point is not globally asymptoticly stable'



Example 4.2. *Consider the difference equation (4.2.1), take* $p = 0.5,\ q = 0.5,\ r = 0.5$. *Equation (4.2.1) becomes*

$$y_{n+1} = \frac{0.5 + 0.5y_{n-1}^2}{1 + y_n + 0.5y_{n-1}^2},\ n = 0,\ 1,\ 2,\ \ldots$$

With initial conditions $y_0 = 0.1,\ y_1 = 1.1$.

The theoretical positive equilibrium point will be $\bar{y} = 0.3926467817$.

Theoretically the positive equilibrium point $\bar{y}$ *is stable since* $pr = 0.25 \leq q$ *and* $q \leq \frac{\sqrt{r}}{\sqrt{2}} = 0.5$.

Figure (4.4) shows that the positive equilibrium point is stable.

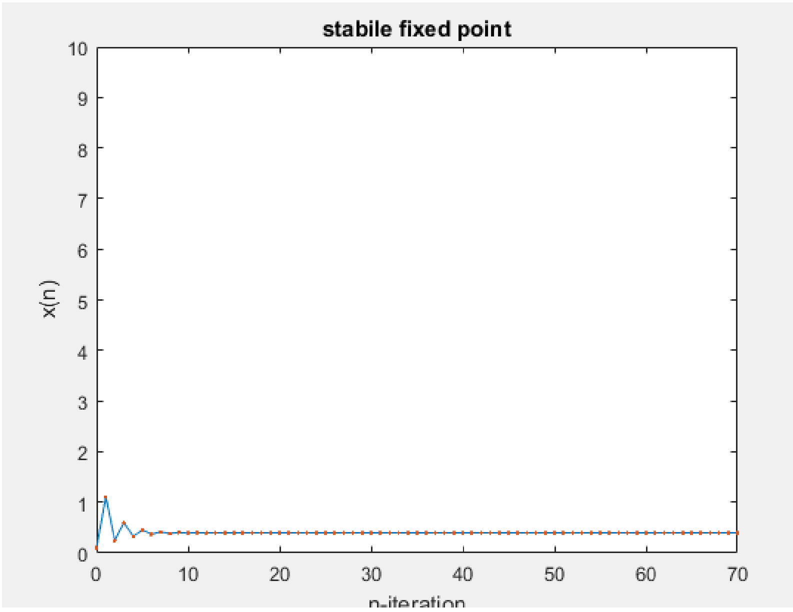


Fig. 4.4: The positive equilibrium point is stable.

And using the Matlab code we get the following output

l =

-0.6963 + 1.4359i

-0.6963 - 1.4359i

0.3926 + 0.0000i

ans = 'The positive fixed point is'

y = 0.3926

a = 0.2672

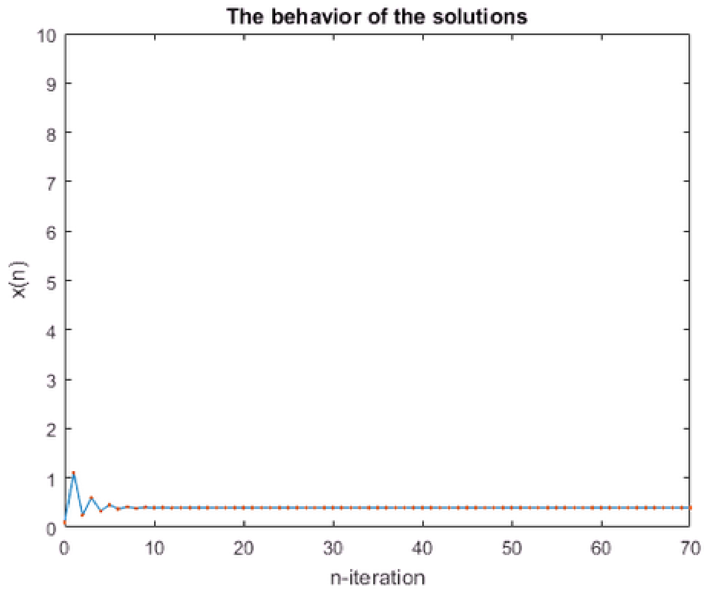
b = 0.1623

ans = 'The positive fixed point is asymptoticly stable'

c = 0.2500

m = 0.5000

ans = 'The positive fixed point is globally asymptoticly stable'


The behavior of the solutions
x(n)
n-iteration
0
1
2
3
4
5
6
7
8
9
10
0
10
20
30
40
50
60
70

4.10 Bifurcation of $y_{n+1} = \frac{p+qy^2{}_{n-1}}{1+y_n+ry_{n-1}^2}$

In this section we study the types of bifurcation that occur at $q = q^*$ as q is the bifurcation parameter.

In order to convert Equation (4.2.1) to a second dimensional system with three parameters p, q, and r, let

$$z_n = y_{n-1},$$

and

$$v_n = y_n.$$

We get the following system

$$z_{n+1} = v_n$$

$$v_{n+1} = \frac{p + qz^2{}_n}{1 + v_n + rz^2{}_n}, \quad n = 0,\ 1,\ 2,\ \ldots$$

This system has the unique fixed point $(\bar{z}, \bar{v})^T = (\bar{y}, \bar{y})^T$. Convert this system in to second dimensional map

$$F\begin{pmatrix} z \\ v \end{pmatrix} = \begin{pmatrix} f_1(z,v) \\ f_2(z,v) \end{pmatrix} = \begin{pmatrix} v \\ \frac{p+qz^2}{1+v+rz^2} \end{pmatrix}. \tag{4.10.1}$$

Now we find the Jacobian matrix of $F(z, v)$, we have

$$\frac{\partial f_1}{\partial z} = 0,$$

$$\frac{\partial f_1}{\partial v} = 1,$$

$$\frac{\partial f_2}{\partial z} = \frac{2qz(1 + v + rz^2) - 2rz(p + qz^2)}{(1 + v + rz^2)^2},$$

and

$$\frac{\partial f_2}{\partial v} = \frac{-(p + qz^2)}{(1 + v + rz^2)^2}.$$

The Jacobian matrix is

$$JF(z, v) = \begin{pmatrix} 0 & 1 \\ \frac{2qz(1+v+rz^2)-2rz(p+qz^2)}{(1+v+rz^2)^2} & \frac{-(p+qz^2)}{(1+v+rz^2)^2} \end{pmatrix}$$

$$JF(z,v)|_{(\bar{y},\bar{y})} = \begin{pmatrix} 0 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}$$

So

$$\det(JF(\bar{y},\bar{y})) = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2},$$

and

$$tr(JF(\bar{y},\bar{y})) = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2}.$$

Theorem 4.4. *The fixed point $(\bar{y},\bar{y})$ of the system (4.10.1) undergoes a saddle-node bifurcation when $q = \frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}$.*

Proof: Saddle-node bifurcation happens when

$$\det(J) = tr(J) - 1.$$

So the fixed point $(\bar{y},\bar{y})$ of the system (4.10.1) undergoes a saddle-node bifurcation if

$$\det(JF(\bar{y},\bar{y})) = tr(JF(\bar{y},\bar{y})) - 1$$

or

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} - 1$$

so

$$-2\bar{y}(q-r\bar{y} = -\bar{y} - 1 - \bar{y} - r\bar{y}^2$$

thus

$$q = \frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}.$$

So saddle-node bifurcation happens if $q = \frac{3r\bar{y}^2+2\bar{y}+1}{2\bar{y}}$. ■

Theorem 4.5. *The fixed point $(\bar{y},\bar{y})$ of the system (4.10.1) undergoes a period-doubling bifurcation when $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$.*

Proof: Period-doubling bifurcation happens when

$$\det(J) = -tr(J) - 1.$$

So the fixed point $(\bar{y}, \bar{y})$ of the system (4.10.1) undergoes a period-doubling bifurcation if

$$\det(JF(\bar{y}, \bar{y})) = -tr(JF(\bar{y}, \bar{y})) - 1$$

or

$$-\frac{2\bar{y}(q - r\bar{y})}{1 + \bar{y} + r\bar{y}^2} = \frac{\bar{y}}{1 + \bar{y} + r\bar{y}^2} - 1$$

so

$$-2\bar{y}(q - r\bar{y}) = \bar{y} - 1 - \bar{y} - r\bar{y}^2$$

thus

$$q = \frac{3r\bar{y}^2 + 1}{2\bar{y}}.$$

So period-doubling bifurcation happens if $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$. ■

Theorem 4.6. *The fixed point* $(\bar{y}, \bar{y})$ *of the system (4.10.1) undergoes Neimark-Sacker bifurcation when* $q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$, *if* $r > \frac{1+\bar{y}}{\bar{y}^2}$.

Proof: Assume $r > \frac{1+\bar{y}}{\bar{y}^2}$. Neimark-Sacker bifurcation which happens when

$$\det(J) = 1$$

and

$$-2 < tr(J) < 2.$$

So the system (4.10.1) undergoes Neimark-Sacker bifurcation when

$$\det(JF(\bar{y}, \bar{y})) = 1 \tag{4.10.2}$$

and

$$-2 < tr(JF(\bar{y}, \bar{y})) < 2.$$

The last inequality always holds, since it is equivalent to

$$-2 < \frac{-\bar{y}}{1 + \bar{y} + r\bar{y}^2} < 2,$$

which can be split to

$$-2 < \frac{-\bar{y}}{1 + \bar{y} + r\bar{y}^2},$$

and

$$\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2,$$

so

$$-2-2\bar{y}-2r\bar{y}^2 < -\bar{y},$$

which implies

$$-2-\bar{y}-2r\bar{y}^2 < 0,$$

Which always holds. And $\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} < 2$ implies

$$2+3\bar{y}+2r\bar{y}^2 > 0.$$

Which also always holds.

Now Equation (4.10.2) holds if

$$-\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 1$$

so

$$-2\bar{y}(q-r\bar{y}) = 1+\bar{y}+r\bar{y}^2$$

thus

$$q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}.$$

Which is positive since $r > \frac{1+\bar{y}}{\bar{y}^2}$. So the system (4.10.1) undergoes Neimark-Sacker bifurcation at $(\bar{y}, \bar{y})$ when $q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$.. ■

4.11 Direction of The Period-Doubling (Flip) bifurcation

In this section we will find the direction of Flip bifurcation of system (4.10.1) at $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$.

We need at first to shift the fixed point $(\bar{y}, \bar{y})$ to the origin. Let

$$w_n = z_n - \bar{y},\ u_n = v_n - \bar{y}.$$

System (4.10.1) will be

$$w_{n+1} = u_n$$
$$u_{n+1} = \frac{p+q(w_n+\bar{y})^2}{1+(u_n+\bar{y})+r(w_n+\bar{y})^2}, \; n = 0, \; 1, \; 2, \; \ldots$$

Or

$$Y_{n+1} = AY_n + G(Y_n), \tag{4.11.1}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}, \; Y_n = \begin{pmatrix} w_n \\ u_n \end{pmatrix},$$

and

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\|Y\|^4)$$
$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \end{pmatrix} \; \textit{and} \; C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \end{pmatrix}$$

where

$$B_i(x,y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j)$$

and

$$C_i(x,y,z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j).$$

So $B_1(\psi,\phi) = 0$ and $C_1(\psi,\phi,\xi) = 0$,

$$B_2(\psi,\phi) = \frac{2q(1+\bar{y}) - 2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_1) - \frac{2\bar{y}(2r\bar{y}+q)}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_2+\psi_2\phi_1) + \frac{2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_2\phi_2),$$

and

$$C_2(\psi,\phi,\xi) = \frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_1) + \frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_2+\psi_1\phi_2\xi_1+\psi_2\phi_1\xi_1) + \frac{4\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_1+\psi_2\phi_1\xi_2+\psi_1\phi_2\xi_2) + \frac{-6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_2).$$

Now we find the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ at the bifurcation point $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$.

Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ respectively. So we have

$$A\hat{q} = -\hat{q}, \textit{ and } A^T p^* = -p^*.$$

Or

$$(A+I)\hat{q} = 0 \tag{4.11.2}$$

$$(A^T+I)p^* = 0. \tag{4.11.3}$$

Equation (4.11.2) is equivalent to

$$\begin{pmatrix} 1 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & 1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\hat{q}_1 = 1$, from the first equation we get

$$\hat{q}_1 + \hat{q}_2 = 0$$

so $\hat{q}_2 = -1$. Thus take $\hat{q} \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Note that in order to have a nonzero solution for $(A+I)\hat{q} = 0$, the matrix $(A+I)$ must be singular. Which means that $|A+I|$ must equal zero, so

$$\frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} + 1 - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0.$$

Thus, $\hat{q}$ satisfies the second equation $\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} - 1 = 0$.

Now, consider Equation (4.11.3) which is equivalent to

$$\begin{pmatrix} 1 & \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} \\ 1 & 1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix} \begin{pmatrix} {p^*}_1 \\ {p^*}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Take ${p^*}_2 = 1$, from the first equation we get

$$ {p^*}_1 + \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}{p^*}_2 = 0 $$

so ${p^*}_1 = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}$. Thus take $p^* \sim \begin{pmatrix} \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} \\ 1 \end{pmatrix}$.

Note that in order to have a nonzero solution for $(A^T + I)p^* = 0$, the matrix $(A^T + I)$ must be singular. Which means that $|A^T + I|$ must equal to zero, so

$$ 1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0. $$

Thus, p^* satisfies the second equation ${p^*}_1 + (1 + (1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}){p^*}_2 = 0$.

Now, we normalize p^* and $\hat{q}$,

$$ \langle p^*, \hat{q} \rangle = \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} - 1. $$

Take $\hat{p} = \eta \begin{pmatrix} \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} \\ 1 \end{pmatrix}$, $\eta = -\frac{1+\bar{y}+r\bar{y}^2}{1+(2q+1)\bar{y}-r\bar{y}^2}$.

The critical eigenspace T^c corresponding to $\lambda = -1$ is one-dimensional and spanned by an eigenvector $\hat{q}$. Let T^{su} denote a one-dimensional linear eigenspace of A corresponding to all eigenvalues other than λ. Note that the matrix $(A - \lambda I_n)$ has common invariant spaces with the matrix A, so we conclude that $y \in T^{su}$ if and only if $\langle \hat{p}, y \rangle = 0$.

Any vector $x \in \mathbb{R}^n$ can be decomposed as

$$ x = u\hat{q} + y $$

where $u\hat{q} \in T^c$, $y \in T^{su}$ and

$$ u = \langle \hat{p}, x \rangle. $$

$$ y = x - \langle \hat{p}, x \rangle \hat{q}. $$

In the coordinates (u, y), the map (4.11.1) can be written as

$$ \tilde{u} = \lambda u + \langle \hat{p}, G(u\hat{q} + y) \rangle, $$

$$\tilde{y} = Ay + G(u\hat{q} + y) - \langle \hat{p}, G(u\hat{q} + y) \rangle \hat{q}.$$

Using Taylor expansions, the last two equations can be written as

$$\begin{aligned} \tilde{u} &= \lambda u + \frac{1}{2}\delta u^2 + u\langle b, y\rangle + \frac{1}{6}\sigma u^3 + ... \\ \tilde{y} &= Ay + \frac{1}{2}au^2 + ... \end{aligned} \tag{4.11.4}$$

where $u = \mathbb{R},\ y \in \mathbb{R}^n,\ \delta,\ \sigma \in \mathbb{R},\ a,\ b \in \mathbb{R}^n$ and $\langle b, y\rangle = \sum_{i=1}^{n} b_i y_i$ is the standard scalar product, and can be expressed as

$$\langle b, y\rangle = \langle b, B(\hat{q}, y)\rangle.$$

The center manifold of (4.11.4) has the representation

$$y = V(u) = \frac{1}{2}w_2 u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset \mathbb{R}^n$, so that $\langle \hat{p}, w_2\rangle = 0$. The vector w_2 satisfies

$$(A - I_n)w_2 + a = 0.$$

We have $\lambda = 1$ is not an eigenvalue of A, so the matrix $(A - I_n)$ is invertible in $\mathbb{R}^n$. Thus, we have

$$w_2 = -(A - I_n)^{-1}a$$

and the restriction of (4.11.4) to the center manifold takes the form

$$\tilde{u} = -u + \frac{1}{2}\delta u^2 + \frac{1}{6}(\sigma - 3\langle \hat{q}, (A - I_n)^{-1}a\rangle)u^3 + O(u^4)$$

where $\delta = \langle \hat{p}, B(\hat{q}, \hat{q})\rangle$, $\sigma = \langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q})\rangle$ and $a = B(\hat{q}, \hat{q}) - \langle \hat{p}, B(\hat{q}, \hat{q})\rangle\hat{q}$.

Using the identity

$$(A - I_n)\hat{q} = \frac{1}{2}\hat{q},$$

the restricted map can be written as

$$\tilde{u} = -u + a(0)u^2 + b(0)u^3 + O(u^4) \tag{4.11.5}$$

where

$$a(0) = \frac{1}{2}\langle \hat{p}, B(\hat{q}, \hat{q})\rangle,$$

and

$$b(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q})\rangle - \frac{1}{4}(\langle \hat{p}, B(\hat{q}, \hat{q})\rangle)^2 - \frac{1}{2}\langle \hat{p}, B(\hat{q}, (A - I_n)^{-1}B(\hat{q}, \hat{q}))\rangle.$$

The map (4.11.5) can be transformed to the normal form

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4)$$

where

$$c(0) = a^2(0) - b(0).$$

Thus,the critical normal form coefficient $c(0)$, allows us to predict the direction of bifurcation of the period-two cycle. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6}\langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q})\rangle - \frac{1}{2}\langle \hat{p}, B(\hat{q}, (A - I_n)^{-1}B(\hat{q}, \hat{q}))\rangle.$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q = \frac{3r\bar{y}^2+1}{2\bar{y}}$.

$$B(\hat{q}, \hat{q}) = \begin{pmatrix} 0 \\ \frac{2\bar{y}(3q+1+4r\bar{y})+2q-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

$$C(\hat{q}, \hat{q}, \hat{q}) = \begin{pmatrix} 0 \\ \frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3} - 3\frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{12\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3} \end{pmatrix}.$$

$$\langle \hat{p}, C(\hat{q}, \hat{q}, \hat{q})\rangle = -\left(\frac{1+\bar{y}+r\bar{y}^2}{1+(2q+1)\bar{y}-r\bar{y}^2}\right)\left[\frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3} - 3\frac{-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{12\bar{y}(q-3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3} + \frac{6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}\right].$$

$$(A - I)^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & -1 + \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}^{-1} = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} -1 + \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} & -1 \\ -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & -1 \end{pmatrix}.$$

$$(A-I)^{-1}B(\hat{q},\hat{q}) = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} \frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \\ \frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

$$B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q})) = \frac{1+\bar{y}+r\bar{y}^2}{2\bar{y}+r\bar{y}^2}\begin{pmatrix} 0 \\ m \end{pmatrix},$$

where

$$m = \left(\frac{-2\bar{y}(3q+1+4r\bar{y})-2q+2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}\right)\times \left(\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}\right).$$

$$\langle \hat{p}, B(\hat{q},(A-I_n)^{-1}B(\hat{q},\hat{q}))\rangle = \left(\left[\frac{2\bar{y}(3q+1+4r\bar{y})+2q-2r(p+2\bar{y}(2q\bar{y}+2r\bar{y}^2))}{(2\bar{y}+r\bar{y}^2)(1+(2q+1)\bar{y}-r\bar{y}^2)}\right] \left[\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}\right]\right).$$

4.12 Direction and stability of Neimark-Sacker bifurcation

To determine the direction of the invariant closed curve that bifurcates from the positive fixed point we will follow the normal form theory of Neimark- Sacker bifurcation given in [5].

Theorem 4.7. *If $q = q^* = \frac{r\bar{y}^2-1-\bar{y}}{2\bar{y}}$, and $r > \frac{1+\bar{y}}{\bar{y}^2}$, then the characteristic equation of (4.2.1) has two complex conjugate roots that lie on the unit circle. Moreover the Neimark-Sacker bifurcation conditions are satisfied if*

$$r < \sqrt{\frac{30\bar{y}(1+\bar{y})^3 + 26\bar{y}(1+\bar{y})^2 + 18\bar{y}^2(1+\bar{y})^2 + 4\bar{y}^3 + 2\bar{y}^4 + 2\bar{y}^2}{2\bar{y}^6}}.$$

Proof: At the beginning we will show that the characteristic equation of (4.2.1)

$$\lambda^2 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\lambda - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0. \qquad (4.12.1)$$

has two complex roots. The roots of (4.12.1) are

$$\lambda_{1,2} = \frac{-\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \pm \sqrt{\Delta}}{2},$$

where

$$\Delta = \frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} + 4\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

Substituting $q = q^*$ we get

$$\Delta = \frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} + 4\frac{-1-\bar{y}-r\bar{y}^2}{1+\bar{y}+r\bar{y}^2}.$$

So

$$\Delta = \frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} - 4.$$

Thus, (4.12.1) has two complex roots if $\Delta < 0$, which is equivlent to

$$\frac{\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^2} - 4 < 0,$$

which implies

$$\bar{y}^2 < 4(1+\bar{y}+r\bar{y}^2)^2,$$

so

$$4(1+2\bar{y}+\bar{y}2+2(1+\bar{y})r\bar{y}^2+r^2\bar{y}^4)-\bar{y}^2 > 0,$$

thus, $\Delta(q^*) < 0$ if

$$4+8\bar{y}+3\bar{y}2+8(1+\bar{y})r\bar{y}^2+4r^2\bar{y}^4 > 0,$$

which always holds.

Next we show that (4.12.1) has two conjugate complex roots on the unit circle when $q = q^*$.

Since $\lambda_{1,2}$ are the roots of (4.12.1) we have

$$\lambda_1\lambda_2 = -\frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2}.$$

Substituting $q = q^*$ we get

$$\lambda_1\lambda_2 = 1.$$

But $\lambda_1\lambda_2 = |\lambda_{1,2}|^2 = 1$. Thus, the two complex roots are on the unit circle.

Assume the roots of (4.12.1) at $q = q^*$ are $e^{\pm i\theta}$, so we have

$$e^{i\theta}+e^{-i\theta} = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2},$$

but $e^{i\theta}+e^{-i\theta} = 2\cos(\theta)$. Thus,

$$\cos(\theta) = -\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

Note that $\frac{-1}{2} < \cos(\theta) < 0$, so there exists $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that

$$\theta_0 = \cos^{-1}\left(-\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}\right).$$

And $e^{ik\theta_0} \neq 1$ for $k = 1,\ 2,\ 3,\ 4$.

Next we will show that $\frac{d|\lambda|^2}{dq}|_{q=q^*} \neq 0$.

$$|\lambda|^2 = -\frac{2\bar{y}(q - r\bar{y})}{1 + \bar{y} + r\bar{y}^2},$$

differentiate with respect to q we get

$$\frac{d|\lambda|^2}{dq} = -\frac{(1 + \bar{y} + r\bar{y}^2)(2\bar{y}(1 - r\frac{d\bar{y}}{dq}) + (q - r\bar{y})2\frac{d\bar{y}}{dq}) - (2\bar{y}(q - r\bar{y}))(\frac{d\bar{y}}{dq} + 2r\bar{y}\frac{d\bar{y}}{dq})}{(1 + \bar{y} + r\bar{y}^2)^2}.$$

To find $\frac{d\bar{y}}{dq}$ we differentiate equation (4.2.2) with respect to q

$$\frac{d}{dq}(r\bar{y}^3 + (1 - q)\bar{y}^2 + \bar{y} - p) = 0,$$

so

$$3r\bar{y}^2\frac{d\bar{y}}{dq} + (1 - q)2\bar{y}\frac{d\bar{y}}{dq} + \bar{y}^2(-1) + \frac{d\bar{y}}{dq} = 0,$$

thus,

$$\frac{d\bar{y}}{dq} = \frac{\bar{y}^2}{3r\bar{y}^2 + (1 - q)2\bar{y} + 1}.$$

Substituting $q = q^*$ we get

$$\frac{d\bar{y}}{dq} = \frac{\bar{y}^2}{2r\bar{y}^2 + 2\bar{y} + 2}.$$

So

$$\frac{d|\lambda|^2}{dq}|_{q=q^*} = -\frac{A(r)}{(1 + \bar{y} + r\bar{y}^2)^2},$$

where

$$A(r) = -10\bar{y}^7r^3 + (-18\bar{y}^5(1 + \bar{y}) - 4\bar{y}^5 + 2\bar{y}^6)r^2 + (-14\bar{y}3(1 + \bar{y})^2 - 4\bar{y}^3(1 + \bar{y}) - \bar{y}^5 - 2\bar{y}^4)r$$
$$-6\bar{y}(1 + \bar{y})^3 - \bar{y}^4 - \bar{y}^3.$$

Since $r > \frac{1+\bar{y}}{\bar{y}^2}$ we have

$$A(r) < 2\bar{y}^6r^2 - (30\bar{y}(1 + \bar{y})^3 + 26\bar{y}(1 + \bar{y})^2 + 18\bar{y}^2(1 + \bar{y})^2 + 4\bar{y}^3 + 2\bar{y}^4 + 2\bar{y}^2).$$

S0 $A(r) < 0$ if

$$r < \sqrt{\frac{30\bar{y}(1 + \bar{y})^3 + 26\bar{y}(1 + \bar{y})^2 + 18\bar{y}^2(1 + \bar{y})^2 + 4\bar{y}^3 + 2\bar{y}^4 + 2\bar{y}^2}{2\bar{y}^6}}.$$

■

We need at first to shift the fixed point $(\bar{y}, \bar{y})$ to the origin. Let

$$w_n = z_n - \bar{y}, \; u_n = v_n - \bar{y}.$$

System (4.10.1) will be

$$w_{n+1} = u_n$$
$$u_{n+1} = \frac{p + q(w_n + \bar{y})^2}{1 + (u_n + \bar{y}) + r(w_n + \bar{y})^2}, \; n = 0, \; 1, \; 2, \; ...$$

Or

$$Y_{n+1} = AY_n + G(Y_n), \tag{4.12.2}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & \frac{-\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}, \; Y_n = \begin{pmatrix} w_n \\ u_n \end{pmatrix},$$

and

$$G(Y) = \frac{1}{2}B(Y, Y) + \frac{1}{6}C(Y, Y, Y) + O(\|Y\|^4)$$
$$B(Y, Y) = \begin{pmatrix} B_1(Y, Y) \\ B_2(Y, Y) \end{pmatrix} \; \text{and} \; C(Y, Y, Y) = \begin{pmatrix} C_1(Y, Y, Y) \\ C_2(Y, Y, Y) \end{pmatrix}$$

where

$$B_i(x, y) = \sum_{k,j=1}^{n} \frac{\partial^2 Y_i(\eta)}{\partial \eta_k \partial \eta_j}|_{\eta=0}(x_k y_j)$$

and

$$C_i(x, y, z) = \sum_{l,k,j=1}^{n} \frac{\partial^3 Y_i(\eta)}{\partial \eta_l \partial \eta_k \partial \eta_j}|_{\eta=0}(x_l y_k z_j).$$

So $B_1(\psi, \phi) = 0$ and $C_1(\psi, \phi, \xi) = 0$,

$$B_2(\psi, \phi) = \frac{2q(1+\bar{y}) - 2r(p + 2\bar{y}(2q\bar{y} - 2r\bar{y}^2))}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_1) -$$
$$\frac{2\bar{y}(2r\bar{y}+q)}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_1\phi_2 + \psi_2\phi_1) + \frac{2\bar{y}}{(1+\bar{y}+r\bar{y}^2)^2}(\psi_2\phi_2),$$

and

$$C_2(\psi, \phi, \xi) = \frac{12r\bar{y}(-3(q(1+\bar{y}) - rp) + 4\bar{y}^2(q - r\bar{y}))}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_1) +$$
$$\frac{-2q(1+\bar{y}) + 24(q - r\bar{y}) - 6rq\bar{y}^2}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_1\phi_1\xi_2 + \psi_1\phi_2\xi_1 + \psi_2\phi_1\xi_1) +$$
$$\frac{4\bar{y}(q - 3r\bar{y})}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_1 + \psi_2\phi_1\xi_2 + \psi_1\phi_2\xi_2) + \frac{-6\bar{y}}{(1+\bar{y}+r\bar{y}^2)^3}(\psi_2\phi_2\xi_2).$$

Now we find the eigenvectors of A and A^T corresponding to the eigenvalue $e^{\pm i\theta_0}$ at the bifurcation point $q = \frac{r\bar{y}^2-\bar{y}-1}{2\bar{y}}$.

Let $\hat{q}$ and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $e^{\pm i\theta_0}$ respectively. So we have

$$A\hat{q} = e^{i\theta_0}\hat{q}, \text{ and } A^T p^* = e^{-i\theta_0}p^*.$$

Or

$$(A - e^{i\theta_0}I)\hat{q} = 0 \tag{4.12.3}$$

$$(A^T - e^{-i\theta_0}I)p^* = 0. \tag{4.12.4}$$

Equation (4.12.3) is equivalent to

$$\begin{pmatrix} -e^{i\theta_0} & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & -e^{i\theta_0} - \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\hat{q}_1 = 1$, from the first equation we get

$$-e^{i\theta_0}\hat{q}_1 + \hat{q}_2 = 0$$

so $\hat{q}_2 = e^{i\theta_0}$. Thus take $\hat{q} \sim \begin{pmatrix} 1 \\ e^{i\theta_0} \end{pmatrix}$.

Note that in order to have a nonzero solution for $(A + I)\hat{q} = 0$, the matrix $(A - e^{i\theta_0}I)$ must be singular. Which means that $|A - e^{i\theta_0}I|$ must equal to zero, so

$$e^{2i\theta_0} + e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0.$$

Thus, $\hat{q}$ satisfies the second equation.

Now, consider Equation (4.12.4) which is equivalent to

$$\begin{pmatrix} -e^{-i\theta_0} & \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} \\ 1 & -e^{-i\theta_0} - \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix} \begin{pmatrix} {p^*}_1 \\ {p^*}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Take ${p^*}_2 = e^{i\theta_0}$, from the second equation we get

$$ {p^*}_1 - 1 - e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} = 0 $$

so ${p^*}_1 = 1 + e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}$. Thus take $p^* \sim \begin{pmatrix} 1 + e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \\ e^{i\theta_0} \end{pmatrix}$.

Note that in order to have a nonzero solution for $(A^T - e^{i\theta_0}I)p^* = 0$, the matrix $(A^T - e^{i\theta_0}I)$ must be singular. Which means that $|A^T - e^{i\theta_0}I|$ must equal to zero, so

$$ e^{-2i\theta_0} + e^{-i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} - \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} = 0. $$

Thus, p^* satisfies the first equation $e^{-i\theta_0}{p^*}_1 + (-e^{-i\theta_0} - \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}){p^*}_2 = 0$.

Now, we normalize p^* and $\hat{q}$,

$$ \langle p^*, \hat{q}\rangle = 2 + \frac{e^{-i\theta_0}\bar{y}}{1+\bar{y}+r\bar{y}^2}. $$

Take $\hat{p} = \eta \begin{pmatrix} 1 + e^{i\theta_0}\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \\ e^{i\theta_0} \end{pmatrix}$, $\eta = \frac{1}{2+\frac{e^{-i\theta_0}\bar{y}}{1+\bar{y}+r\bar{y}^2}}$.

The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and spanned by $\{Re(\hat{q}), Im(\hat{q})\}$. The real space T^s corresponding to real eigenvalues of A is $(n-2)$-dimensional. $y \in T^{su}$ if and only if $\langle \hat{p}, y\rangle = 0$. Note that $y \in \mathbb{R}^n$ is real, while $\hat{p} \in \mathbb{C}^n$.

Any vector $x \in \mathbb{R}^n$ can be decomposed as

$$ x = z\hat{q} + \bar{z}\bar{\hat{q}} + y $$

where $z \in \mathbb{C}^1$, $z\hat{q} + \bar{z}\bar{\hat{q}} \in T^c$ and $y \in T^s$. The complex variable z is a coordinate on T^c. We have

$$ z = \langle \hat{p}, x\rangle $$
$$ y = x - \langle \hat{p}, x\rangle\hat{q} - \langle \bar{\hat{p}}, x\rangle\bar{\hat{q}} $$

In these coordinates,the map (4.12.2) takes the form

$$\tilde{z} = e^{i\theta_0}z + \langle \hat{p}, G(z\hat{q} + \bar{z}\bar{\hat{q}} + y)\rangle$$
$$\tilde{y} = Ay + G(z\hat{q} + \bar{z}\bar{\hat{q}} + y) - \langle \hat{p}, G(z\hat{q} + \bar{z}\bar{\hat{q}} + y)\rangle\hat{q} - \langle \bar{\hat{p}}, G(z\hat{q} + \bar{z}\bar{\hat{q}} + y)\rangle\bar{\hat{q}}.$$

The previous system can be written as

$$\tilde{z} = e^{i\theta_0}z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} + \langle G_{10}, y\rangle z + \langle G_{10}, y\rangle \bar{z}$$
$$\tilde{y} = Ay + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \frac{1}{2}H_{21}z^2\bar{z},$$

where $G_{20},\ G_{11},\ G_{02},\ G_{21} \in \mathbb{C}^1$ and $G_{01},\ G_{10},\ H_{ij} \in \mathbb{C}^n$ and the scaler product is in $\mathbb{C}^n$.

The complex numbers and vectors can be computed by

$$G_{20} = \langle \hat{p}, B(\hat{q}, \hat{q})\rangle,\ G_{11} = \langle \hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle,\ G_{02} = \langle \hat{p}, B(\bar{\hat{q}}, \bar{\hat{q}})\rangle,\ \ G_{21} = \langle \hat{p}, C(\hat{q}, \hat{q}, \bar{\hat{q}})\rangle$$

and

$$H_{20} = B(\hat{q}, \hat{q}) - \langle \hat{p}, B(\hat{q}, \hat{q})\rangle\hat{q} - \langle \bar{\hat{p}}, B(\hat{q}, \hat{q})\rangle\bar{\hat{q}},$$
$$H_{11} = B(\hat{q}, \bar{\hat{q}}) - \langle \hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle\hat{q} - \langle \bar{\hat{p}}, B(\hat{q}, \bar{\hat{q}})\rangle\bar{\hat{q}},$$

and

$$\langle G_{10}, y\rangle = \langle \hat{p}, B(\hat{q}, y)\rangle,\ \langle G_{01}, y\rangle = \langle \hat{p}, B(\bar{\hat{q}}, y)\rangle.$$

The center manifold in the previous system has the representation

$$Y = V(z, \bar{z}) = \frac{1}{2}v_{20}z^2 + v_{11}z\bar{z} + \frac{1}{2}v_{02}\bar{z}^2$$

where $\langle \hat{q}, v_{ij}\rangle = 0$. The vectors $v_{ij} \in \mathbb{C}^n$ can be found from linear equations

$$(e^{2i\theta_0}I - A)v_{20} = H_{20},$$

$$(I - A)v_{11} = H_{11},$$

$$(e^{-2i\theta_0}I - A)v_{02} = H_{02}.$$

These equations have unique solutions. The matrix $(I - A)$ is invertible because 1 is not an eigenvalue of A. If

$$e^{3i\theta_0} \neq 1$$

the matrices $(e^{\pm 2i\theta_0}I - A)$ are also invertible in $\mathbb{C}^n$ because $e^{\pm 2i\theta_0}$ are not eigenvalues of A. Thus, generically the restricted map can be written as

$$\tilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}[G_{21} + 2\langle\bar{\hat{p}}, B(\hat{q}, (I-A)^{-1}H_{11})\rangle + \langle\bar{\hat{p}}, B(\bar{\hat{q}}, (e^{2i\theta_0}I - A)^{-1}H_{20})\rangle]z^2\bar{z} + ...$$

taking into account the identities

$$(I-A)^{-1}\hat{q} = \frac{1}{1-e^{i\theta_0}}\hat{q},\ (e^{2i\theta_0}I - A)^{-1}\hat{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\hat{q},\ (I-A)^{-1}\bar{\hat{q}} = \frac{1}{1-e^{i\theta_0}}\bar{\hat{q}},$$

and

$$(e^{2i\theta_0}I - A)^{-1}\bar{\hat{q}} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\bar{\hat{q}}.$$

Also z can be written using the map

$$\tilde{z} = e^{i\theta_0}z + \sum_{k,l\geq 2}\frac{1}{k!j!}g_{kj}z^k\bar{z}^j, \tag{4.12.5}$$

where $g_{20} = \langle\hat{p}, B(\hat{q}, \hat{q})\rangle$, $g_{11} = \langle\hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle$, $g_{02} = \langle\hat{p}, B(\bar{\hat{q}}, \bar{\hat{q}})\rangle$, and

$$g_{21} = \langle\hat{p}, C(\hat{q}, \hat{q}, \bar{\hat{q}})\rangle + 2\langle\hat{p}, B(\hat{q}, (I-A)^{-1}B(\hat{q}, \bar{\hat{q}}))\rangle + \langle\hat{p}, B(\bar{\hat{q}}, (e^{2i\theta_0}I - A)^{-1}B(\hat{q}, \hat{q}))\rangle + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{-i\theta_0}}\langle\hat{p}, B(\hat{q}, \hat{q})\rangle\langle\hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle - \frac{2}{1-e^{-i\theta_0}}|\langle\hat{p}, B(\hat{q}, \bar{\hat{q}})\rangle|^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1}|\langle\hat{p}, B(\bar{\hat{q}}, \bar{\hat{q}})\rangle|^2.$$

Or equivalently

$$\bar{z} = e^{i\theta_0}z(1 + d(0))|z^4|$$

where $a(0) = Re\{d(0)\}$, that determines the direction of bifurcation of a closed invariant curve, can be computed by formula

$$a(0) = Re\left(\frac{e^{-i\theta_0}g_{21}}{2}\right) - Re\left(\frac{(1-e^{2i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2.$$

Where

$$g_{20} = \langle\hat{p}, B(\hat{q}, \hat{q})\rangle.$$

$$B(\hat{q}, \hat{q}) = \begin{pmatrix} 0 \\ \frac{2q(1+\bar{y}) - 2r(p + 2\bar{y}(2q\bar{y} - 2r\bar{y}^2)) - 4\bar{y}(2r\bar{y}+q)e^{i\theta_0} + 2\bar{y}e^{2i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

$$\langle \hat{p}, B(\hat{q},\hat{q})\rangle = \frac{1}{2+\frac{e^{-i\theta_0}\bar{y}}{1+\bar{y}+r\bar{y}^2}} e^{i\theta_0}[\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2}].$$

So

$$g_{20} = \frac{e^{i\theta_0}(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0})}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

$$g_{11} = \langle \hat{p}, B(\hat{q},\bar{\hat{q}})\rangle,$$

$$B(\hat{q},\bar{\hat{q}}) = \begin{pmatrix} 0 \\ \frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))+2\bar{y}-4\bar{y}(2r\bar{y}+q)\cos(\theta_0)}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix} = \begin{pmatrix} 0 \\ m \end{pmatrix}.$$

$$\langle \hat{p}, B(\hat{q},\bar{\hat{q}})\rangle = \frac{1}{2+\frac{e^{-i\theta_0}\bar{y}}{1+\bar{y}+r\bar{y}^2}} e^{i\theta_0}\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))+2\bar{y}-4\bar{y}(2r\bar{y}+q)\cos(\theta_0)}{(1+\bar{y}+r\bar{y}^2)^2}.$$

So

$$g_{11} = \frac{e^{i\theta_0}[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))+2\bar{y}-4\bar{y}(2r\bar{y}+q)\cos(\theta_0)]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

$$g_{02} = \langle \hat{p}, B(\bar{\hat{q}},\bar{\hat{q}})\rangle,$$

$$B(\bar{\hat{q}},\bar{\hat{q}}) = \begin{pmatrix} 0 \\ \frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{-i\theta_0}+2\bar{y}e^{-2i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix},$$

$$\langle \hat{p}, B(\bar{\hat{q}},\bar{\hat{q}})\rangle = \frac{1}{2+\frac{e^{-i\theta_0}\bar{y}}{1+\bar{y}+r\bar{y}^2}} e^{i\theta_0}\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{-i\theta_0}+2\bar{y}e^{-2i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2}.$$

So

$$g_{02} = \frac{e^{i\theta_0}[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{-i\theta_0}+2\bar{y}e^{-2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

Now to find g_{21}

$$g_{21} = \langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle + 2\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle +$$
$$\langle \hat{p}, B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{-i\theta_0}}\langle \hat{p}, B(\hat{q},\hat{q})\rangle\langle \hat{p}, B(\hat{q},\bar{\hat{q}})\rangle$$
$$-\frac{2}{1-e^{-i\theta_0}}|\langle \hat{p}, B(\hat{q},\bar{\hat{q}})\rangle|^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1}|\langle \hat{p}, B(\bar{\hat{q}},\bar{\hat{q}})\rangle|^2.$$

$$C(\hat{q},\hat{q},\bar{\hat{q}}) = \begin{pmatrix} 0 \\ \frac{12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))+(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0)+e^{i\theta_0})+4\bar{y}(q-3r\bar{y})(2+e^{2i\theta_0})-6\bar{y}e^{i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^3} \end{pmatrix}.$$

So

$$\langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle = \frac{e^{i\theta_0}[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2} +$$
$$\frac{e^{i\theta_0}[(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0)+e^{i\theta_0})+4\bar{y}(q-3r\bar{y})(2+e^{2i\theta_0})-6\bar{y}e^{i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}.$$

And

$$\begin{aligned}(I-A)^{-1} &= \begin{pmatrix} 1 & -1 \\ \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & 1+\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2}\end{pmatrix}^{-1} \\ &= \frac{1+\bar{y}+r\bar{y}^2}{1+2\bar{y}-2q\bar{y}+3r\bar{y}^2}\begin{pmatrix} 1+\frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & 1\end{pmatrix}\end{aligned}$$

$$(I-A)^{-1}B(\hat{q},\bar{\hat{q}}) = \frac{1+\bar{y}+r\bar{y}^2}{1+2\bar{y}-2q\bar{y}+3r\bar{y}^2}\begin{pmatrix} m \\ m\end{pmatrix} = \begin{pmatrix} s \\ s\end{pmatrix},$$

where

$$s = \frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))+2\bar{y}-4\bar{y}(2r\bar{y}+q)\cos(\theta_0)}{(1+2\bar{y}-2q\bar{y}+3r\bar{y}^2)(1+\bar{y}+r\bar{y}^2)}.$$

$$B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}})) = \begin{pmatrix} 0 \\ M\end{pmatrix}$$

where

$$M = s\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}(2r\bar{y}+q)(1+e^{i\theta_0})+2\bar{y}e^{i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2}.$$

So

$$\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle = \frac{e^{i\theta_0}M(1+\bar{y}+r\bar{y}^2)}{2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y}}.$$

Finally,

$$(e^{2i\theta_0}I - A)^{-1} = \begin{pmatrix} e^{2i\theta_0} & -1 \\ \frac{-2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & e^{2i\theta_0} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} \end{pmatrix}^{-1}$$

$$= \frac{1+\bar{y}+r\bar{y}^2}{e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y})} \begin{pmatrix} e^{2i\theta_0} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}^2} & 1 \\ \frac{2\bar{y}(q-r\bar{y})}{1+\bar{y}+r\bar{y}^2} & e^{2i\theta_0} \end{pmatrix}$$

$$(e^{2i\theta_0}I - A)^{-1}B(\hat{q},\hat{q}) =$$

$$\frac{1+\bar{y}+r\bar{y}^2}{e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y})} \begin{pmatrix} \frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2} \\ e^{2i\theta_0}\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}$$

$$= \begin{pmatrix} L \\ e^{2i\theta_0}L \end{pmatrix}$$

where

$$L = \frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y}))(1+\bar{y}+r\bar{y}^2)}.$$

$$B(\bar{\hat{q}}, (e^{2i\theta_0}I - A)^{-1}B(\hat{q},\hat{q})) = L \begin{pmatrix} 0 \\ \frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}(2r\bar{y}+q)(e^{2i\theta_0}+e^{-i\theta_0})+2\bar{y}e^{2i\theta_0}}{(1+\bar{y}+r\bar{y}^2)^2} \end{pmatrix}.$$

So

$$\langle \hat{p}, B(\bar{\hat{q}}, (e^{2i\theta_0}I - A)^{-1}B(\hat{q},\hat{q}))\rangle = \frac{e^{i\theta_0}L[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)} + \frac{e^{i\theta_0}L[-2\bar{y}(2r\bar{y}+q)(e^{2i\theta_0}+e^{-i\theta_0})+2\bar{y}e^{2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}.$$

$$a(0) = \frac{1}{2}Re\left(e^{-i\theta_0}\langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle\right) + Re\left(e^{-i\theta_0}\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle\right) + \frac{1}{2}Re\left(e^{-i\theta_0}\langle \hat{p}, B(\bar{\hat{q}},(e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle\right).$$

Let

$$B_1 = Re\left(e^{-i\theta_0}\langle \hat{p}, C(\hat{q},\hat{q},\bar{\hat{q}})\rangle\right),\ B_2 = Re\left(e^{-i\theta_0}\langle \hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle\right),$$

and

$$B_3 = Re\left(e^{-i\theta_0}\langle \hat{p}, B(\bar{\hat{q}}, (e^{2i\theta_0}I - A)^{-1}B(\hat{q}, \hat{q}))\rangle\right).$$

To find B_1:

$$e^{-i\theta_0}\langle \hat{p}, C(\hat{q}, \hat{q}, \bar{\hat{q}})\rangle = \frac{[12r\bar{y}(-3(q(1+\bar{y}) - rp) + 4\bar{y}^2(q - r\bar{y}))]}{(2(1+\bar{y}+r\bar{y}^2) + e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2} +$$
$$\frac{e^{i\theta_0}[(-2q(1+\bar{y}) + 24(q - r\bar{y}) - 6rq\bar{y}^2)(\cos(\theta_0) + e^{i\theta_0}) + 4\bar{y}(q - 3r\bar{y})(2 + e^{2i\theta_0}) - 6\bar{y}e^{i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2) + e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)^2}$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$(4(1+\bar{y}+r\bar{y}^2)^2 + 2(1+\bar{y}+r\bar{y}^2)\bar{y}(e^{-i\theta_0} + e^{i\theta_0}) + \bar{y}^2)(1+\bar{y}+r\bar{y}^2)^2,$$

which equals,

$$(4(1+\bar{y}+r\bar{y}^2)^2 + 4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0) + \bar{y}^2)(1+\bar{y}+r\bar{y}^2)^2 = A_1.$$

Multiplying the numerator by the conjugate of the complex part of the denominator, we get,

$$\begin{gathered}
2(1+\bar{y}+r\bar{y}^2)[12r\bar{y}(-3(q(1+\bar{y}) - rp) + 4\bar{y}^2(q - r\bar{y}))] + \\
\bar{y}[12r\bar{y}(-3(q(1+\bar{y}) - rp) + 4\bar{y}^2(q - r\bar{y}))]e^{i\theta_0} \\
+2(1+\bar{y}+r\bar{y}^2)(-2q(1+\bar{y}) + 24(q - r\bar{y}) - 6rq\bar{y}^2)(\cos(\theta_0)) + \\
\bar{y}e^{i\theta_0}(-2q(1+\bar{y}) + 24(q - r\bar{y}) - 6rq\bar{y}^2)(\cos(\theta_0)) + \\
2(1+\bar{y}+r\bar{y}^2)(-2q(1+\bar{y}) + 24(q - r\bar{y}) - 6rq\bar{y}^2)(e^{i\theta_0}) + \\
e^{2i\theta_0}\bar{y}(-2q(1+\bar{y}) + 24(q - r\bar{y}) - 6rq\bar{y}^2) + \\
16(1+\bar{y}+r\bar{y}^2)\bar{y}(q - 3r\bar{y}) + 8(1+\bar{y}+r\bar{y}^2)\bar{y}(q - 3r\bar{y})(e^{2i\theta_0}) + \\
8\bar{y}^2(q - 3r\bar{y})(e^{i\theta_0}) + 4\bar{y}^2(q - 3r\bar{y})(e^{3i\theta_0}) - 12\bar{y}(1+\bar{y}+r\bar{y}^2)e^{i\theta_0} \\
-6\bar{y}^2e^{2i\theta_0}.
\end{gathered}$$

Taking the real part of the numerator, we get

$$2(1+\bar{y}+r\bar{y}^2)[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]+$$
$$\bar{y}[12r\bar{y}(-3(q(1+\bar{y})-rp)+4\bar{y}^2(q-r\bar{y}))]\cos(\theta_0)$$
$$+4(1+\bar{y}+r\bar{y}^2)(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos(\theta_0))+$$
$$\bar{y}(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)(\cos^2(\theta_0))+$$
$$\bar{y}(-2q(1+\bar{y})+24(q-r\bar{y})-6rq\bar{y}^2)\cos(2\theta_0)+$$
$$16(1+\bar{y}+r\bar{y}^2)\bar{y}(q-3r\bar{y})+8(1+\bar{y}+r\bar{y}^2)\bar{y}(q-3r\bar{y})(\cos(2\theta_0))+$$
$$8\bar{y}^2(q-3r\bar{y})(\cos(\theta_0))+4\bar{y}^2(q-3r\bar{y})(\cos(3\theta_0))-12\bar{y}(1+\bar{y}+r\bar{y}^2)\cos(\theta_0)$$
$$-6\bar{y}^2\cos(2\theta_0)=A_2.$$

So $B_1 = \frac{A_2}{A_1}$.

To find B_2

$$e^{-i\theta_0}\langle\hat{p}, B(\hat{q},(I-A)^{-1}B(\hat{q},\bar{\hat{q}}))\rangle = s\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-2\bar{y}(2r\bar{y}+q)(1+e^{i\theta_0})}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}$$
$$+\frac{2\bar{y}e^{i\theta_0}}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$(4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2)=A_3.$$

Multiplying the numerator by the conjugate of the complex part of the denominator, we get,

$$s[2(1+\bar{y}+r\bar{y}^2)(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)))$$
$$+\bar{y}(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)))e^{i\theta_0}-4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q)$$
$$-4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q)(e^{i\theta_0})-2\bar{y}^2(2r\bar{y}+q)e^{i\theta_0}$$
$$-2\bar{y}^2(2r\bar{y}+q)(e^{2i\theta_0})+4\bar{y}(1+\bar{y}+r\bar{y}^2)e^{i\theta_0}+2\bar{y}^2e^{2i\theta_0}].$$

Taking the real part of the numerator, we get

$$s[2(1+\bar{y}+r\bar{y}^2)(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)))$$
$$+\bar{y}(2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2)))\cos(\theta_0)-4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q)$$
$$-4(1+\bar{y}+r\bar{y}^2)\bar{y}(2r\bar{y}+q)\cos(\theta_0)-2\bar{y}^2(2r\bar{y}+q)\cos(\theta_0)$$
$$-2\bar{y}^2(2r\bar{y}+q)\cos(2\theta_0)+4\bar{y}(1+\bar{y}+r\bar{y}^2)\cos(\theta_0)+2\bar{y}^2\cos(2\theta_0)]=A_4.$$

So $B_2 = \frac{A_4}{A_3}$.

To find B_3

$$e^{-i\theta_0}\langle\hat{p}, B(\bar{\hat{q}}, (e^{2i\theta_0}I-A)^{-1}B(\hat{q},\hat{q}))\rangle =$$
$$\frac{[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}+$$
$$\frac{[-2\bar{y}(2r\bar{y}+q)(e^{2i\theta_0}+e^{-i\theta_0})+2\bar{y}e^{2i\theta_0}]}{(2(1+\bar{y}+r\bar{y}^2)+e^{-i\theta_0}\bar{y})(1+\bar{y}+r\bar{y}^2)}$$
$$\times\frac{2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))-4\bar{y}(2r\bar{y}+q)e^{i\theta_0}+2\bar{y}e^{2i\theta_0}}{(e^{4i\theta_0}(1+\bar{y}+r\bar{y}^2)+e^{2i\theta_0}\bar{y}-2\bar{y}(q-r\bar{y}))(1+\bar{y}+r\bar{y}^2)}.$$

Multiplying and dividing by the conjugate of the complex part of the denominator, the denominator becomes,

$$(4(1+\bar{y}+r\bar{y}^2)^2+4(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(\theta_0)+\bar{y}^2)(1+\bar{y}+r\bar{y}^2)^2[(1+\bar{y}+r\bar{y}^2)^2+\bar{y}^2$$
$$+4\bar{y}^2(q-r\bar{y})^2+2(1+\bar{y}+r\bar{y}^2)\bar{y}\cos(2\theta_0)$$
$$-4\bar{y}(q-r\bar{y})(1+\bar{y}+r\bar{y}^2)\cos(4\theta_0)-4\bar{y}(q-r\bar{y})\cos(2\theta_0)]=A_5.$$

Multiplying the numerator by the conjugate of the complex part of the denominator, we get,

$$[(a_1+a_6+a_{12})+(a_2+a_7+a_{10})e^{i\theta_0}+(a_3+a_4+a_{11})e^{2i\theta_0}+a_5e^{3i\theta_0}+a_8e^{4i\theta_0}+a_9e^{-i\theta_0}]\times$$
$$(b_1e^{-4i\theta_0}+b_2e^{-2i\theta_0}+b_3+b_4e^{-3i\theta_0}+b_5e^{-i\theta_0}+b_6e^{i\theta_0})$$
$$=(a_1+a_6+a_{12})b_1e^{-4i\theta_0}+(a_1+a_6+a_{12})b_2e^{-2i\theta_0}+(a_1+a_6+a_{12})b_3+$$
$$(a_1+a_6+a_{12})b_4e^{-3i\theta_0}+(a_1+a_6+a_{12})b_5e^{-i\theta_0}+(a_1+a_6+a_{12})b_6e^{i\theta_0}+$$
$$(a_2+a_7+a_{10})b_1e^{-3i\theta_0}+(a_2+a_7+a_{10})b_2e^{-i\theta_0}+(a_2+a_7+a_{10})b_3e^{i\theta_0}+$$
$$(a_2+a_7+a_{10})b_4e^{-2i\theta_0}+(a_2+a_7+a_{10})b_5+(a_2+a_7+a_{10})b_6e^{2i\theta_0}+$$
$$(a_3+a_4+a_{11})b_1e^{-2i\theta_0}+(a_3+a_4+a_{11})b_2+(a_3+a_4+a_{11})b_3e^{2i\theta_0}+$$
$$(a_3+a_4+a_{11})b_4e^{-i\theta_0}+(a_3+a_4+a_{11})b_5e^{i\theta_0}+(a_3+a_4+a_{11})b_6e^{3i\theta_0}+$$
$$a_5b_1e^{-i\theta_0}+a_5b_2e^{i\theta_0}+a_5b_3e^{3i\theta_0}+a_5b_4+a_5b_5e^{i\theta_0}+a_5b_6e^{4i\theta_0}+$$
$$a_8b_1+a_8b_2e^{2i\theta_0}+a_8b_3e^{4i\theta_0}+a_8b_4e^{i\theta_0}+a_8b_5e^{3i\theta_0}+$$
$$a_8b_6e^{5i\theta_0}+a_9b_1e^{-5i\theta_0}+a_9b_2e^{-3i\theta_0}+a_9b_3e^{-i\theta_0}+$$
$$a_9b_4e^{-4i\theta_0}+a_9b_5e^{-2i\theta_0}+a_9b_6$$

Where

$$a_1=[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]^2.$$
$$a_2=-4\bar{y}(2r\bar{y}+q)[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_3=2[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]\bar{y}.$$
$$a_4=-2\bar{y}(2r\bar{y}+q)[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_5=a_6=8\bar{y}^2(2r\bar{y}+q)^2,\quad a_7=a_8=-4\bar{y}^2(2r\bar{y}+q).$$

$$a_9=-2[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))]\bar{y}(2r\bar{y}+q).$$
$$a_{10}=2\bar{y}[2q(1+\bar{y})-2r(p+2\bar{y}(2q\bar{y}-2r\bar{y}^2))].$$

$$a_{11}=-8\bar{y}^2(2r\bar{y}+q),\quad a_{12}=4\bar{y}^2.$$

$$b_1 = 2(1+\bar{y}+r\bar{y}^2)^2, \quad b_2 = 2(1+\bar{y}+r\bar{y}^2)\bar{y}.$$

$$b_3 = -4(1+\bar{y}+r\bar{y}^2)\bar{y}(q-r\bar{y}), \quad b_4 = (1+\bar{y}+r\bar{y}^2)\bar{y}.$$

$$b_5 = \bar{y}^2, \quad b_6 = -2\bar{y}^2(q-r\bar{y}).$$

Taking the real part of the numerator, we get

$$\begin{gathered}
(a_8b_6 + a_9b_1)\cos(5\theta_0) + ((a_1+a_6+a_{12})b_1 + a_9b_4 + a_8b_3 + a_5b_6)\cos(4\theta_0) + \\
((a_1+a_6+a_{12})b_4 + (a_2+a_7+a_{10})b_1 + (a_3+a_4+a_{11})b_6 + a_5b_3 + a_8b_5 + a_9b_2)\cos(3\theta_0) + \\
((a_1+a_6+a_{12})b_2 + (a_2+a_7+a_{10})b_4 + (a_2+a_7+a_{10})b_6 + (a_3+a_4+a_{11})b_1 \\
+(a_3+a_4+a_{11})b_3 + a_8b_2 + a_9b_5)\cos(2\theta_0) + ((a_1+a_6+a_{12})b_5 \\
+(a_1+a_6+a_{12})b_6 + (a_2+a_7+a_{10})b_2 + (a_2+a_7+a_{10})b_3 + (a_3+a_4+a_{11})b_4 \\
+(a_3+a_4+a_{11})b_5 + a_5b_1 + a_5b_2 + a_5b_5 + a_8b_4 + a_9b_3)\cos(\theta_0) \\
+(a_1+a_6+a_{12})b_3 + (a_2+a_7+a_{10})b_5 + (a_3+a_4+a_{11})b_2 + a_5b_4 + a_8b_1 + a_9b_6 \\
= A_6.
\end{gathered}$$

Where

$$\cos(\theta_0) = -\frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

$$\cos(2\theta_0) = 2\cos^2(\theta_0) - 1 = \frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1.$$

$$\cos(3\theta_0) = 4\cos^3(\theta_0) - 3\cos(\theta_0) = -\frac{\bar{y}^3}{2(1+\bar{y}+r\bar{y}^2)^3} + \frac{3\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

$$\cos(4\theta_0) = 2\cos^2(2\theta_0) - 1 = 2\left(\frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1\right)^2 - 1.$$

$$\cos(5\theta_0) = 2\cos(2\theta_0)\cos(3\theta_0) - \cos(\theta_0)$$

$$= 2\left(\frac{\bar{y}^2}{2(1+\bar{y}+r\bar{y}^2)^2} - 1\right)\left(-\frac{\bar{y}^3}{2(1+\bar{y}+r\bar{y}^2)^3} + \frac{3\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}\right) + \frac{\bar{y}}{2(1+\bar{y}+r\bar{y}^2)}.$$

SO $B_3 = \frac{A_6}{A_5}$. And

$$a(0) = \frac{1}{2}B_1 + B_2 + \frac{1}{2}B_3.$$

Theorem 4.8. *If $a(0) < 0$ (respectively,> 0), then Neimark-Saker bifurcation of system (4.12.2) at $q = q^*$ is supercritical (respectively, subcritical) and there exists a unique invariant closed curve bifurcates from the positive fixed point $\bar{y}$ which is asymptotically stable (respectively, unstable).*

4.13 Matlab codes and numerical discussion 2

In this section we introduce Matlab codes that use our results, and then we insert some examples.

Matlab code for period-doubling bifurcation:

```
r=  ; %r value
p=  ; %p value
u=0;
a=-r/2
t = [a 1 0.5 -p];
l= roots(t)
for m= 1:3
if l(m)>0
     u=u+1;
 'The positive fixed point is'
y=l(m)
'The bifurcation value of the parameter q is '
q=(3*r*y^2+1)/(2*y)
A=-1*( (1+y+r*(y)^2)/(1+(2*q+1)*y-r*(y)^2));
B=12*r*y*(-3*(q*(1+y)-r*p)+4*y^2*(q-r*y))-3*(-2*q*(1+y)+24*(
    q-r*y)-6*r*q*y^2)+ 12*y*(q-3*r*y) + 6*y;
 U=(1+y+r*y^2)^3 ;
```

```
D=A*(B/U);
F=(2*y*(3*q+1+4*r*y )+2*q-2*r*(p+2*y*(2*q*y+2*r*y^2) ) )
    /((2*y+r*y^2 )*(1+(2*q+1)*y-r*y^2 ) );
L=(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2 ) )-2*y )/((1+y+r*y
    ^2)^2 );
J=F*L;
c=(1/6)*D-(0.5 )*J
if c>0
    'A unique and stable period-two cycle bifurcates from
       the fixed point at the bifurcation point.'
end
end
end
amin=0;
amax=10;
x0=.2;x1=.3;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
x(i+1)=(p+a.*(x(i-1))^2)/(1+(x(i))+r*(x(i-1))^2);
if (i>750)
z(j,i-750)=x(i+1);
end
```

```
end
end
plot(t,z,'blue','MarkerSize',5),title ('Period-doubling
    bifurcation')
if u==0
    'The system does not undergo period-doubling (flip )
        bifurcation. '
end
```

Example 4.3. *Consider the difference equation (4.2.1). Fix p, r, and consider q as bifurcation parameter. Take $p = 0.5$, $r = 1.8$, and $0 < q \leq 10$. Equation (4.2.1) becomes*

$$y_{n+1} = \frac{0.5 + qy_{n-1}^2}{1 + y_n + 1.8y_{n-1}^2}, \quad n = 0, \ 1, \ 2, \ \ldots \tag{4.13.1}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{0.5+qy_1(n)^2}{1+y_2(n)+1.8y_1(n)^2} \end{pmatrix}. \tag{4.13.2}$$

The positive equilibrium point $\bar{y}$ of (4.13.1) satisfies

$$1.8\bar{y}^3 + (1-q)\bar{y}^2 + \bar{y} - 0.5 = 0. \tag{4.13.3}$$

Theorem 4.2 shows that the fixed point undergoes a period-doubling bifurcation at $q^* = \frac{3\times 1.8\bar{y}^2+1}{2\bar{y}}$. So Equation (4.13.3) at q^* becomes

$$-0.9\bar{y}^3 + \bar{y}^2 + 0.5\bar{y} - 0.5 = 0.$$

Which has two positive roots, so we have two values of q^*.

Thus the first value of q^* gives the following fixed point of (4.13.1)

$$\bar{y} = 0.6495.$$

Substituting the value of $\bar{y}$ in q^* we get

$$q^* = 2.5235.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$c(0) = 0.9539 > 0$$

So this shows that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^* = 2.5235$.

The second value of q^* *gives the following fixed point of (4.13.1)*

$$\bar{y} = 1.1840.$$

Substituting the value of $\bar{y}$ *in* q^* *we get*

$$q^* = 3.6192.$$

Now to determine the direction of period-doubling bifurcation we find $c(0)$.

$$c(0) = -0.4132$$

So this shows that no stable period-two cycle bifurcates from the fixed point at the bifurcation point $q^* = 3.6192$.

Figure (4.5) shows the stable period-two cycle.

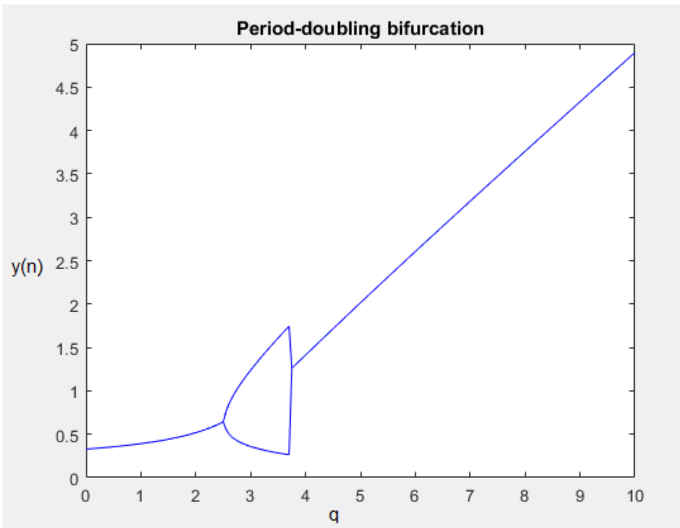


Fig. 4.5: Period-doubling bifurcation of $y_{n+1} = \frac{0.5+qy_{n-1}^2}{1+y_n+1.8y_{n-1}^2}$.

And using the Matlab code we get the following output

a = -0.9000

l =

-0.7224

1.1840

0.6495

ans = 'The positive fixed point is'

y = 1.1840

ans = 'The bifurcation value of the parameter q is '

q = 3.6192

c = -0.4132

ans = 'The positive fixed point is'

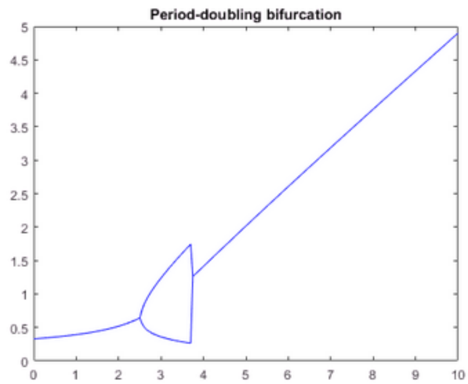
y = 0.6495

ans = 'The bifurcation value of the parameter q is '

q = 2.5235

c = 0.9539

ans = ' A unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point.'



Matlab code for Neimark-Sacker bifurcation:

```
r=  ;%r value
p=  ; %p value
e=r/2;
t = [e 1.5 1.5 -p];
l= roots(t)
i=0;
for m=1:3
if l(m)>0
y=l(m);
if (r*y^2-y-1)/(2*y)>0
    'The positive fixed point is'
y=l(m)
'The bifurcation value of the parameter q is '
q=(r*y^2-y-1)/(2*y)
     i=i+1;
     T=((30*y*(1+y)^3+26*y*(1+y)^2+18*(y^2)*((1+y)^2)+4*y
        ^3+2*y^4+2*y^2)/(2*y^6))^(0.5);
     if r<T
        'The Neimark-Sacker bifurcation conditions hold '

'cos(Theta_0)='
o=-1*(y)/(2*(1+y+r*y^2))
Theta_0=acos(o)
A1=(4*(1+y+r*y^2)^2+4*(1+y+r*y^2)*y*cos(Theta_0)+y^2)*((1+y+
    r*y^2)^2 );
A2=2*(1+y+r*y^2)*(12*r*y*(-3*(q*(1+y)-r*p )+4*y^2*(q-r*y)))+
    y*(12*r*y*(-3*(q*(1+y)-r*p )+4*y^2*(q-r*y)))*cos(Theta_0)
    +4*(1+y+r*y^2)*(-2*q*(1+y)+24*(q-r*y)-6*r*q*y^2)*(cos(
    Theta_0))+y*(-2*q*(1+y)+24*(q-r*y)-6*r*q*y^2)*(cos(
    Theta_0))^2+y*(-2*q*(1+y)+24*(q-r*y)-6*r*q*y^2)*cos(2*
```

```
    Theta_0)+16*(1+y+r*y^2)*y*(q-3*r*y)+8*((1+y+r*y^2))*y*(q
    -3*r*y)*cos(2*Theta_0)+8*y^2*(q-3*r*y)*cos(Theta_0)+4*y
    ^2*(q-3*r*y)*cos(3*Theta_0)-12*y*(1+y+r*y^2)*cos(Theta_0)
    -6*y^2*cos(2*Theta_0);
B1=A2/A1
A3=(4*(1+y+r*y^2)^2+4*(1+y+r*y^2)*y*cos(Theta_0)+y^2)*((1+y+
    r*y^2));
s=(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2 ))+2*y -4*y*(2*r*y+q)
    *cos(Theta_0))/((1+2*y-2*q*y + 3*r*y^2 ) *(1+y+r*y^2)      )
    ;
A4 = s*(2*(1+y+r*y^2)*(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2))
    )+y*(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2)))*cos(Theta_0)
    -4*(1+y+r*y^2)*y*(2*r*y+q)-4*(1+y+r*y^2)*y*(2*r*y+q)*cos(
    Theta_0)-2*y^2*(2*r*y+q)*cos(Theta_0)-2*y^2*(2*r*y+q)*cos
    (2*Theta_0)+4*y*(1+y+r*y^2)*cos(Theta_0)+2*y^2*cos(2*
    Theta_0));
B2=A4/A3
A5=(4*(1+y+r*y^2)^2+4*(1+y+r*y^2)*y*cos(Theta_0)+y^2)*((1+y+
    r*y^2)^2 )*((1+y+r*y^2)^2+y^2+4*y^2*(q-r*y)^2+2*(1+y+r*y
    ^2)*y*cos(2*Theta_0)-4*y*(q-r*y)*(1+y+r*y^2)*cos(4*
    Theta_0)-4*y*(q-r*y)*cos(2*Theta_0)    );
a1=(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2)))^2;
a2=-4*y*(2*r*y+q)*(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2)));
a3=2*(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2)))*y;
a4=-2*y*(2*r*y+q)*(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2)));
a5=8*(y^2)*(2*r*y+q)^2;
a6=8*(y^2)*(2*r*y+q)^2;
a7=-4*(y^2)*(2*r*y+q);
a8=-4*(y^2)*(2*r*y+q);
a9=-2*(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2)))*y*(2*r*y+q);
a10=2*y*(2*q*(1+y)-2*r*(p+2*y*(2*q*y-2*r*y^2)));
```

```
a11=-8*(y^2)*(2*r*y+q);
a12=4*y^2;
b1=2*(1+y+r*y^2)^2;
b2=2*(1+y+r*y^2)*y;
b3=-4*(1+y+r*y^2)*y*(q-r*y );
b4=(1+y+r*y^2)*y;
b5=y^2;
b6=-2*(y^2)*(q-r*y);
A6=(a8*b6 +a9*b1)*cos(5*Theta_0)+((a1 +a6+a12)*b1 +a9*b4+a8*
    b3+a5*b6)*cos(4*Theta_0)+((a1 +a6+a12)*b4+(a2+a7+a10)*b1
    +(a3+a4+a11)*b6+a5*b3+a8*b5+a9*b2)*cos(3*Theta_0)+((a1 +
    a6+a12)*b2+(a2+a7+a10)*b4+(a2+a7+a10)*b6 +(a3+a4+a11)*b1
    +(a3+a4+a11)*b3+a8*b2+ a9*b5 )*cos(2*Theta_0)+((a1 +a6+
    a12)*b5+(a1 +a6+a12)*b6 +(a2+a7+a10)*b2+(a2+a7+a10)*b3+(
    a3+a4+a11)*b4 +(a3+a4+a11)*b5 +a5*b1 +a5*b2 +a5*b5 +a8*b4
     +a9*b3)*cos(Theta_0)+(a1 +a6+a12)*b3+(a2+a7+a10)*b5+(a3+
    a4+a11)*b2+a5*b4+a8*b1+a9*b6;
B3=(A6)/(A5)
a_0=0.5*B1+B2+0.5*B3
if a_0<0
   'the Neimark- Sacker bifurcation is supercritical.'
else
     'the Neimark- Sacker bifurcation is subcritical.'
end
     end
end
end
end
if i==0
    'The system does not undergo Neimark-Sacker bifurcation.
       '
```

```
end
'The bifurcation diagram:'
amin=0;
amax=10;
x0=1;x1=1;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
x(i+1)=(p+a.*(x(i-1))^2)/(1+(x(i))+r*(x(i-1))^2);
    if (i>750)
z(j,i-750)=x(i+1);
end
end
end
plot(t,z,'blue','MarkerSize',6),xlabel('parameter q'),
   ylabel('y(n+1)')
```

Example 4.4. *Consider the difference equation (4.2.1). Fix p, r, and consider q as bifurcation parameter. Take $p = 2$, $r = 9$, and $0 < q \leq 10$. Equation (4.2.1) becomes*

$$y_{n+1} = \frac{2 + qy_{n-1}^2}{1 + y_n + 9y_{n-1}^2}, \ n = 0, \ 1, \ 2, \ \ldots \tag{4.13.4}$$

Which is equivalent to

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{2+qy_1(n)^2}{1+y_2(n)+9y_1(n)^2} \end{pmatrix}. \quad (4.13.5)$$

The positive equilibrium point $\bar{y}$ of (4.13.4) satisfies

$$9\bar{y}^3 + (1-q)\bar{y}^2 + \bar{y} - 2 = 0. \quad (4.13.6)$$

Theorem shows that the fixed point undergoes a Neimark-Sacker bifurcation at $q^* = \frac{9\bar{y}^2-\bar{y}-1}{2\bar{y}}$. So Equation (4.13.6) at q^* becomes

$$4.5\bar{y}^3 + 1.5\bar{y}^2 + 1.5\bar{y} - 2 = 0.$$

Which has one positive roots.

Thus the value of q^* gives the following fixed point of (4.13.4)

$$\bar{y} = 0.5462.$$

Substituting the value of $\bar{y}$ in q^* we get

$$q^* = 1.0424.$$

Now to determine the direction of period-doubling bifurcation we find $a(0)$.

$$a(0) = 11.7658 > 0$$

So this shows that the Neimark- Sacker bifurcation at $q^* = 1.0424$ is subcritical.

Figure (4.6) shows that the positive fixed point $\bar{y}$ is asymptotically stable for $q > q^*$ and change it's stability at Neimark-Sacker bifurcation value q^*. Figures (4.7) and (4.8) shows phase portraits associated with figure (4.6) at q^* and $q = 1.1$, respectively.

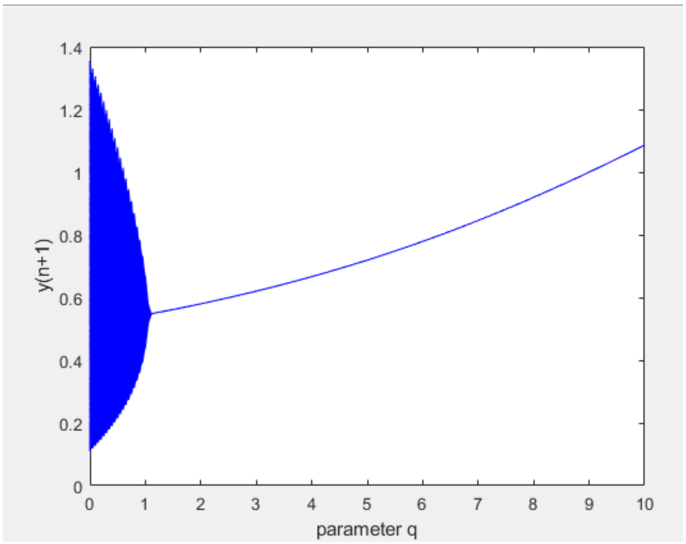


Fig. 4.6: Neimark-Sacker bifurcation of $y_{n+1} = \frac{2+qy_{n-1}^2}{1+y_n+9y_{n-1}^2}$.

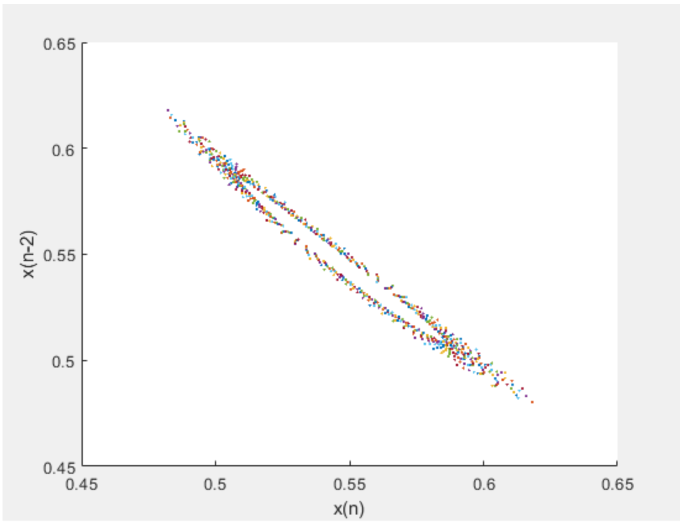


Fig. 4.7: Phase portraits of the map $y_{n+1} = \frac{2+qy_{n-1}^2}{1+y_n+9y_{n-1}^2}$ at q^*.

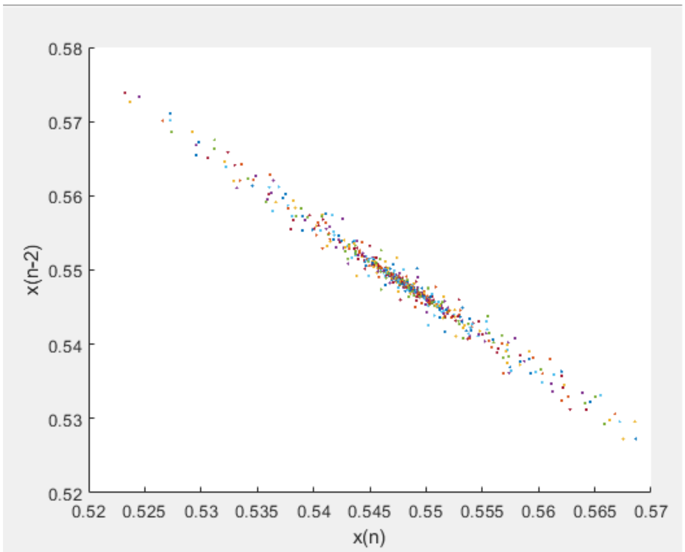


Fig. 4.8: Phase portraits of the map $y_{n+1} = \frac{2+qy_{n-1}^2}{1+y_n+9y_{n-1}^2}$ at $q = 1.1$.

And using the Matlab code we get the following output

l =

-0.4398 + 0.7876i

-0.4398 - 0.7876i

0.5462 + 0.0000i

ans = 'The positive fixed point is'

y = 0.5462

ans = 'The bifurcation value of the parameter q is '

q = 1.0424

ans = 'The Neimark-Sacker bifurcation conditions hold '

ans = '$cos(Theta_0)$ ='

o = -0.0645

$Theta_0$ = 1.6354

B1 = 17.2905

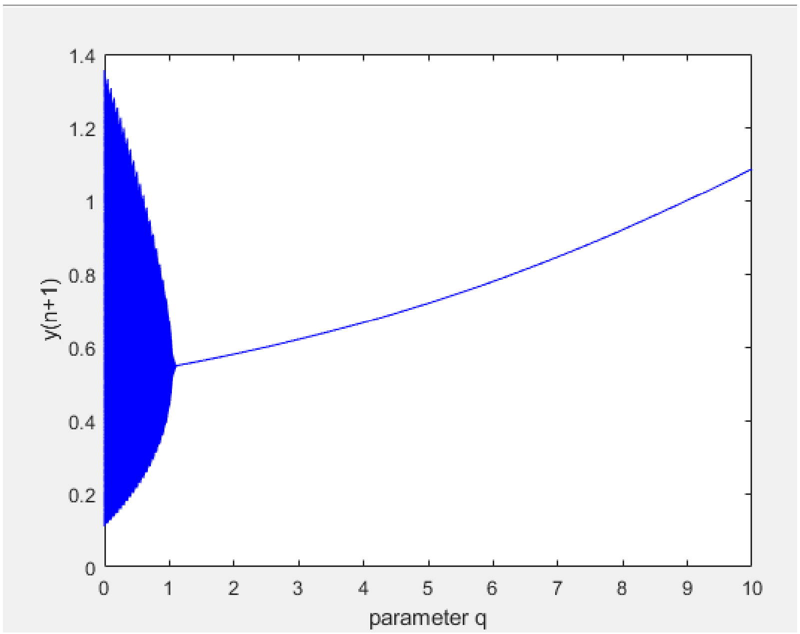
B2 = 1.5522

B3 = 3.1367

$a_0 = 11.7658$

ans = 'the Neimark- Sacker bifurcation is subcritical.'

ans = 'The bifurcation diagram:'



5. CONCLUSION

In this thesis, we considered the second order, quadratic rational difference equations

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n^2 + Cx_{n-1}}, \ n = 0, \ 1, \ 2, \ ...$$

and

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}^2}{A + Bx_n + Cx_{n-1}^2}, \ n = 0, \ 1, \ 2, \ ...$$

With positive parameters $\alpha, \ \beta, \ A, \ B, \ C$, and non-negative initial conditions.
We focused on local stability, invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points. And we studied the types of bifurcation exist where the change of stability occurs. Then, we give some Matlab codes that use thesis results and numerical discussions with figures to support our results.

For the **first equation**, the change of variables

$$x_n = \frac{\sqrt{A}}{\sqrt{B}} y_n.$$

reduced equation (3.0.1) to the difference equation

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + {y^2}_n + ry_{n-1}}, \ n = 0, \ 1, \ 2, \ ...$$

Where $p = \alpha\frac{\sqrt{B}}{\sqrt{A^3}}$, $q = \frac{\beta}{A}$, and $r = \frac{C}{\sqrt{AB}}$.

We proved the existence of the unique positive equilibrium point of our difference equation, and then we gave a Matlab code to find it.

Then we found the linearized equation and the characteristic equation. And we checked when the unique positive equilibrium point $\bar{y}$ of equation (3.2.1) is locally asymptotically stable. We investigated also two invariant intervals. And we showed that any solution takes its values between 0 and $p + \frac{q}{r}$.

Then we set some conditions on q that must hold when two periodic cycle exist. And we gave a case for global stability. And we introduced Matlab code that uses our results for finding the fixed point and its stability and solution behavior, and then we gave some examples.

Finally, we studied the bifurcation of our difference equation. And we concentrated at the Period-Doubling (Flip) Bifurcation and its direction.

And for the **second equation**, the change of variables

$$x_n = \frac{A}{B} y_n.$$

reduced equation (4.0.1) to the difference equation

$$y_{n+1} = \frac{p + q{y^2}_{n-1}}{1 + y_n + r{y^2}_{n-1}}, \quad n = 0,\ 1,\ 2,\ \ldots$$

Where $p = \alpha \frac{B}{A^2}$, $q = \frac{\beta}{B}$, and $r = \frac{CA}{B^2}$.

We proved the existence of the unique positive equilibrium point of our difference equation, and then we insert a Matlab code to find it.

Then we found the linearized equation and the characteristic equation. And we checked when the unique positive equilibrium point $\bar{y}$ of equation (4.0.1) is locally asymptotically stable. We investigated also two invariant intervals. And we show that any solution take its values between 0 and $p + \frac{q}{r}$.

Then we set some conditions on q that must hold when two periodic cycle exist. And we gave a case for global stability. And we introduced Matlab code that uses our results for finding the fixed point and its stability and solution behavior, and then we gave some example.

Finally, we studied the bifurcation of our difference equation. And we concentrated at the Period-Doubling (Flip) Bifurcation and the Neimark-Sacker bifurcation and their directions.

6. APPENDICES

Matlab code for chapter one

Figures 1.2 and 1.3

The Cobweb diagram and the behavior of the solutions near the fixed point of $f(x) = 3x - x^2$.

```
r=3; % growth rate
x0=0.2; %initial population x0
n=80; %end of time interval
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=x0;
tt(1)=0;
for i=1:n
t(i)=i-1;
x(i+1)=x(i)*(r-x(i));
end
t(n+1)=n;
nn=100;
del=3./nn;
xstart=0;
yy=zeros(nn+1,1);
lin=zeros(nn+1,1);
```

```
xx=zeros(nn+1,1);
for i=1:nn+1
xx(i)=xstart+(i-1)*del;
lin(i)=xx(i);
yy(i)=xx(i)*(r-xx(i));
end
plot(xx,lin,xx,yy)
title('cobweb diagram for f(x)=x(2-x), x0=0.2'),pause
xc=zeros(24,1);
yc=zeros(24,1);
xc(1)=x0;
yc(1)=0;
xc(2)=x0;
yc(2)=x0*(r-x0);
yc(3)=yc(2);
xc(3)=yc(2);
plot(xx,lin,xx,yy,xc,yc),axis ([0 2 0 1.5]),pause
for j=3:20;
jj=2*j-4;
xc(jj)=xc(jj-1);
yc(jj)=xc(jj)*(r-xc(jj));
xc(jj+1)=yc(jj);
yc(jj+1)=yc(jj);
plot(xx,lin,xx,yy,xc,yc), axis ([0 2 0 1.5]),pause
end
plot(t,x,t,x,'.')
xlabel('n-iteration'),ylabel('x(n)'), axis ([0 80 0 1.5])
title ('stability of fixed point')
```

Matlab code for chapter two

Figure 2.3

Saddle-node bifurcation of the map $f(x) = x^2 - cx$.

```
figure (8), hold on
for i=-0.25:0.0001:0.75
x=-(1-sqrt(1+4*i))/2;
plot(i,x,'-','MarkerSize',6)
axis([-0.5 1 -2 2])
end
for i=-0.25:0.01:0.75
y=-(1+sqrt(1+4*i))/2;
plot (i,y,'--','MarkerSize',6)
title('Saddle-node bifurcation of the map f(x)=x^2-c x')
xlabel('parameter c'),ylabel('x')
end
hold off
```

Figure 2.4

Transcritical bifurcation of the map $f(x) = x^2 + cx$.

```
figure (6), hold on
for i=-1:0.03:1
x=1-i;
plot(i,x,'--','MarkerSize',6)
axis([-2 4 -2.5 2.5])
end
for i=-1:0.001:1
y=0;
```

```
plot (i,y,'--','MarkerSize',6)
end
for i=1:0.001:3
x=1-i;
plot(i,x,'--','MarkerSize',6)
end
for i=1:0.03:3
y=0;
plot(i,y,'--','MarkerSize',6)
title(Transcritical bifurcation of the map f(x)=x^2+c x)
xlabel('parameter c'),ylabel('x')
end
hold off
```

Figure 2.5

Pitchfork bifurcation of $f(x) = cx - 2x^3$.

```
figure (7), hold on
for i=-1:0.001:1
x=0;
plot (i,x,'MarkerSize',6), axis([-1.2 2.2 -1 1])
end
for i=1:0.001:2
x=-sqrt((i-1)/2);
y=sqrt((i-1)/2);
plot (i,x,'blue',i,y,'blue','MarkerSize',6)
end
for i=1:0.04:2
x=0;
plot(i,x,'-', 'MarkerSize',6)
```

```
xlabel('parameter c')
ylabel('x')
title(Pitchfork bifurcation of f(x)=c x -2x^3)
end
hold off
```

Matlab code for chapter three

Figure 3.1

The behavior of the solutions of $y_{n+1} = \frac{4+5y_{n-1}}{1+y_n^2+0.5y_{n-1}}$.

```
n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(4+5*x(i-1))/(1+(x(i))^2+0.5*x(i-1));
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('unstabile fixed point')
```

Figure 3.2

The behavior of the solutions of $y_{n+1} = \frac{0.5+0.3y_{n-1}}{1+y_n^2+0.5y_{n-1}}$.

```
n=70;
```

```
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(0.5+0.3*x(i-1))/(1+(x(i))^2+0.5*x(i-1));
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('unstabile fixed point')
```

Figure 3.3

Period-doubling bifurcation of $y_{n+1} = \frac{1+qy_{n-1}}{1+y_n^2+0.9y_{n-1}}$.

```
amin=0;
amax=10;
x0=.2;x1=.3;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
x(i+1)=(1+a.*x(i-1))/(1+(x(i))^2+.9.*x(i-1));
```

```
if (i>750)
z(j,i-750)=x(i+1);
end
end
end
plot(t,z,'blue','MarkerSize',5),title ('Period-doubling
    bifurcation')
```

Matlab code for chapter four

Figure 4.3

The behavior of the solutions of $y_{n+1} = \frac{0.4+5y_{n-1}^2}{1+y_n+0.5y_{n-1}^2}$.

```
n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(0.4+5*(x(i-1))^2)/(1+(x(i))+0.5*(x(i-1))^2);
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('unstabile fixed point')
```

Figure 4.4

The behavior of the solutions of $y_{n+1} = \frac{0.5+0.5y_{n-1}^2}{1+y_n+0.5y_{n-1}^2}$.

```
n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(0.5+0.5*(x(i-1))^2)/(1+(x(i))+0.5*(x(i-1))^2);
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
axis ([0 70 0 10]), title ('stabile fixed point')
```

Figure 4.5

Period-doubling bifurcation of $y_{n+1} = \frac{0.5+qy_{n-1}^2}{1+y_n+1.8y_{n-1}^2}$.

```
amin=0;
amax=10;
x0=.2;x1=.3;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
```

```
x(i+1)=(0.5+a.*(x(i-1))^2)/(1+(x(i))+1.*(x(i-1))^2);
if (i>750)
z(j,i-750)=x(i+1);
end
end
end
plot(t,z,'blue','MarkerSize',5),title ('Period-doubling
    bifurcation')
```

Figure 4.6

Neimark-Sacker bifurcation of $y_{n+1} = \frac{2+qy_{n-1}^2}{1+y_n+9y_{n-1}^2}$.

```
amin=0;
amax=10;
x0=1;x1=1;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
x(i+1)=(2+a.*(x(i-1))^2)/(1+(x(i))+9*(x(i-1))^2);
    if (i>750)
z(j,i-750)=x(i+1);
end
```

```
end
end
plot(t,z,'blue','MarkerSize',6),xlabel('parameter q'),
    ylabel('y(n+1)')
```

Figure 4.7

Phase portraits of the map $y_{n+1} = \frac{2+qy_{n-1}^2}{1+y_n+9y_{n-1}^2}$ at q^*.

```
N=1000; x(1)=1;x(2)=1;
for q= 1.0424
for n=2:1:0.3*N
x(n+1)=(2+q.*(x(n-1))^2)/(1+(x(n))+9*(x(n-1))^2);
x(n-1);
end
figure (2), hold on
for n=0.3*N :1:N
x(n+1)=(2+q.*(x(n-1))^2)/(1+(x(n))+9*(x(n-1))^2);
x(n);
plot(x(n),x(n-2),'.','MarkerSize',5),axis([0.45 0.65 0.45
    0.65])
xlabel('x(n)'),ylabel('x(n-2)')
end
end
hold off
```

Figure 4.8

Phase portraits of the map $y_{n+1} = \frac{2+qy_{n-1}^2}{1+y_n+9y_{n-1}^2}$ at $q = 1.1$.

```
N=900; x(1)=1;x(2)=1;
for q=1.1
for n=2:1:0.3*N
x(n+1)=(2+q.*(x(n-1))^2)/(1+(x(n))+9*(x(n-1))^2);
x(n-1);
end
figure (2), hold on
for n=0.3*N :1:N
x(n+1)=(2+q.*(x(n-1))^2)/(1+(x(n))+9*(x(n-1))^2);
x(n);
plot(x(n),x(n-2),'.','MarkerSize',5),axis([0.52 0.57 0.52
    0.58])
xlabel('x(n)'),ylabel('x(n-2)')
end
end
hold off
```

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